2

Computation

Whole number computation—adding, subtracting, multiplying, and dividing—has always been a major topic in the elementary school curriculum. And this focus is still justifiable, even in today's technological age. First, being able to compute is a practical skill that lets us answer questions such as how much? how many? how many times greater? We use computation every day—when verifying that a store clerk has given us the correct change, figuring the tip to leave in a restaurant, or determining how much it will cost to buy the required number of party favors for a birthday celebration. Second, whole number computation is the foundation of arithmetic. It can help students make sense of mathematical relationships and prepare them for later work with fractions, signed numbers, and algebra. Whether computations involve "naked" numbers or are embedded in word problems, students need to know how to solve them. They must be able to determine when to use mental math, paper and pencil, or the calculator. Most important, their methods must make sense, both to themselves and to others.

1. Mathematical Properties

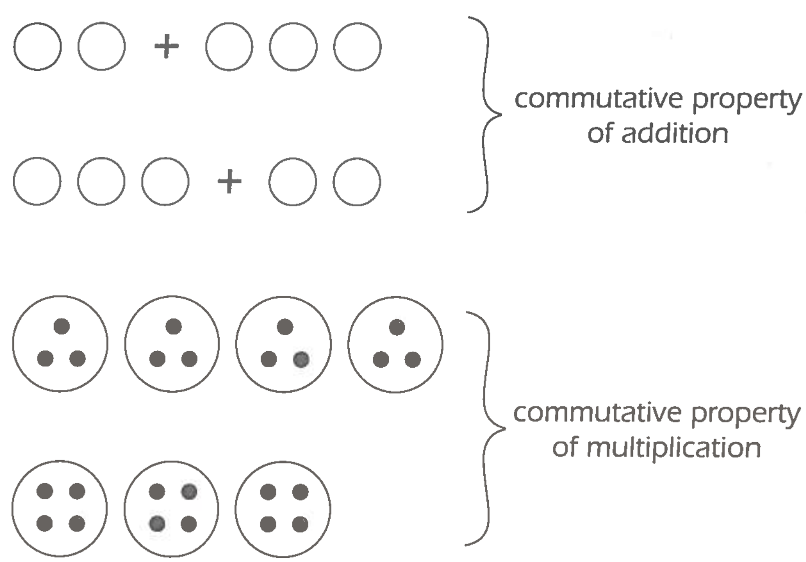
All of us use the commutative, associative, and distributive properties when working with whole numbers, fractions, and decimals. Yet many teachers consider properties to be "too mathematical for young students" and "not especially relevant to what we teach in the elementary grades." The opposite is closer to the truth. These properties, together with the operations of addition, subtraction, multiplication, and division, are the foundation of arithmetic. We use mathematical properties when performing computations and recalling basic facts. We use them to reduce the complexity of equations and expressions and to perform mental calculations more easily. Children often notice the relationships on which these properties are based (e.g., the order of the addends does not affect the sum) and use them when computing. The goal is for elementary students to build upon their understanding of these properties using whole numbers, fractions, and decimals and in the middle grades to be able to represent these properties symbolically with variables. Mathematical properties are an avenue to higher-level thinking, because they illustrate general cases and can lead to mathematical generalizations.

The Commutative Property

The commutative properties of addition and multiplication state that the order of two addends (e.g., $12 + 9$ or $9 + 12$) or two factors (e.g., 3×7 or 7×3) does not affect the sum or product, respectively. The root word of commutative is *commute*, which means to interchange—we can reverse the order of two addends or two factors without changing the result. These powerful properties reduce the number of basic facts students need to memorize: having learned that $5 + 6 = 11$ or $3 \times 4 = 12$, they also know that $6 + 5 = 11$ and $4 \times 3 = 12$.

But not all students recognize on their own that the order of the addends or factors doesn't matter in terms of the sum or product. One reason is that when they first learn basic addition or multiplication, they focus on the operation and on obtaining an answer that ideally makes sense. They don't reflect on the outcome of the operation in relation to the order of the numbers. Often teachers decide to introduce the idea of the commutative property after students have become familiar with either addition or multiplication. They might help students discover that when adding $3 + 9$ in their head, for example, it is easier to start with the 9 and count on three numbers (10, 11, 12) than to start with the 3 and count on nine numbers. However, this still doesn't necessarily mean students will generalize that the order of the addends doesn't matter. Teachers can then present a number of examples of the commutative property of addition ($6 + 8 = 14$ and $8 + 6 = 14$, $23 + 57 = 80$ and $57 + 23 = 80$, etc.), ask students to identify any patterns in the number pairs and make a generalization about the pattern, and then ask them to investigate whether their generalization holds true for other numbers.

The commutative property of multiplication can be especially confusing to students. In an addition operation, the addends represent subgroups comprising the same things: if $2 + 3$ is expressed as $3 + 2$, only the order of the subgroups changes. In the grouping interpretation of multiplication, 3×4 and 4×3 represent different groupings—three groups of four versus four groups of three—and do not look the same.



Students need to model many multiplication equations in order to see that the products are identical. A rectangular array (consisting of rows and columns) often helps students make sense of multiplication's commutative property, because the dimensions of an array can be depicted in different orders (5 by 8 and 8 by 5, for example) but each arrangement consists of the same number of squares. Rotating an array bearing row and column labels also provides a strong visual image of commutativity.

The commutative property of multiplication can also be confusing since switching the factors switches the relationships in some word problems. Sometimes the switched relationships are similar and still make sense; other times they change the problem completely. For example, running 3 miles at 10 minutes a mile is quite different from running 10 miles at 3 minutes per mile, even though total running time in each case is 30 minutes. The first rate is within normal ranges, but someone running 10 miles in 30 minutes would be making history! Instruction therefore also needs to focus on how the relationships in multiplication problems change when the numbers are switched even though the products are still equal.

Activity

How Does Order Affect Differences?

Objective: learn how order affects differences and identify and explain the resulting patterns.

Pick a simple subtraction equation such as $6 - 2 = 4$. Now reverse the two numbers and perform the new subtraction, $2 - 6 = {}^{-}4$. Pick another equation and reverse the numbers. Do this a number of times. Examine the pairs of differences (e.g., 4 and $^{-}4$). What patterns do you see in the differences? Why does this pattern occur?

Things to Think About

Students are told often that the order of the numbers doesn't matter in addition but does in subtraction. While it is true that most of the time order does matter in subtraction, there is an exception. When both numbers are identical, order doesn't matter ($20 - 20 = 20 - 20$).

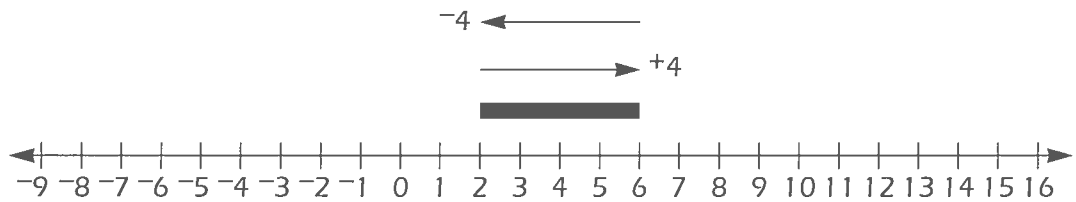
Even though subtraction is not commutative, there are interesting relationships in the differences of numbers when their order is reversed for subtraction. When the two numbers are not the same (e.g., 2 and 6), the pair of differences are opposites: one difference is a positive number (e.g., 4) and the other is its negative mirror image (e.g., $^{-}4$). This can be represented formally as: $a - b = c$ and $b - a = {}^{-}c$ if $a \neq b \neq 0$. This pattern holds for fractions, decimals, and integers, as shown in the examples below:

$$\frac{5}{8} - \frac{2}{8} = \frac{3}{8} \qquad 1.4 - 1.2 = 0.2 \qquad 7 - ({}^{-}4) = 11$$

$$\frac{2}{8} - \frac{5}{8} = \frac{{}^{-}3}{8} \qquad 1.2 - 1.4 = {}^{-}0.2 \qquad {}^{-}4 - 7 = {}^{-}11$$

Did you notice that the pairs of differences sum to zero (e.g., ${}^{+}4 + {}^{-}4 = 0$)? The number 0 is the identity element for addition. You can add 0 to any other number and the sum will be the original number. Why does this pattern of opposites occur when we change the order of the numbers in a subtraction problem? The difference between two numbers can be represented on a number line as an interval—the distance between two points. Since a distance is something we can measure, the interval is either a positive value or zero. Thus the distance between

2 and 6 is 4. But we can also think of this interval in terms of a direction (see Chapter 1, page 22). If we start at 2 and move to the right toward the positive numbers, we represent this action as adding a positive 4 and we land at 6. However, if we start at 6 and move to the left toward the negative numbers, this action is represented as adding a negative 4.



Interestingly, once we include negative numbers in the set of numbers with which we are working (for example, the real number set), the operation of subtraction is technically unnecessary. Instead of subtracting, we can obtain the same answer by adding the opposite of the subtrahend (the number we are taking away). One reason we refer to relationships as "additive" instead of "additive and subtractive" is because in the set of real numbers, by using opposites, it is possible to rewrite all subtraction expressions and equations as equivalent addition expressions and equations. For example, we can solve $10 - 4$ by adding the opposite of 4, or $^{-}4$ ($10 + {}^{-}4$) and we can solve $10 - (-2)$ by adding the opposite of $^{-}2$, or 2 ($10 + 2$). Examine the steps below that show why these pairs of expressions ($10 - 4$ and $10 + {}^{-}4$; and $10 - {}^{-}2$ and $10 + 2$) are equivalent.

$10 - 4$	$10 - {}^{-}2$	
$10 + ({}^{-}4 + 4) - 4$	$10 + (2 + {}^{-}2) - {}^{-}2$	Add 0
$10 + {}^{-}4 + (4 - 4)$	$10 + 2 + ({}^{-}2 - {}^{-}2)$	Associative property
$10 + {}^{-}4 + 0$	$10 + 2 + 0$	Add 0
$10 + {}^{-}4$	$10 + 2$	

Students whose instruction has focused on examining differences on number lines or hundred charts sometimes use this understanding of differences in their mental calculations. For example, one technique for computing $34 - 19$ involves thinking of both numbers in terms of tens and ones ($30 + 4$ and $10 + 9$). The numbers in each place are subtracted separately without regrouping ($30 - 10$ and $4 - 9$), with the ones place resulting in a negative value. The tens and ones values are then combined—$20 + (-5)$—for an answer of 15.

34	*Think:* $30 - 10 = 20$
$-\ 19$	$4 - 9 = {}^{-}5$
20	
$+\ (-5)$	
15	$20 + (-5) = 20 - 5 = 15$

Young students' explanations rarely acknowledge that they are using negative numbers, but they often comment that $4 - 9$ is "5 below zero" or they might mention that they are "5 in the hole" or "owe 5." They may either subtract the 5 from the 20 or add $^{-}5$ to 20. Teachers are encouraged to revisit these ideas with older students as a way of expanding their understanding of properties and operations. ▲

Activity

2

How Does Order Affect Quotients?

Objective: explore patterns in the quotient when the dividend and divisor are reserved.

Pick a simple division equation such as $10 \div 2 = 5$. The number 10 is the *dividend*, 2 is the *divisor*, and 5 is the *quotient*, or answer. Next reverse the dividend and the divisor and perform the new division, $2 \div 10 = \frac{1}{5}$. Pick another equation and reverse the dividend and divisor. Do this a number of times. Examine the pairs of quotients (e.g., 5 and $\frac{1}{5}$). What patterns do you see in the different quotients? Do these patterns hold for all types of numbers?

Things to Think About

Division is also not commutative, but did you notice that when you reverse the order of the dividend and the divisor, the resulting quotients are reciprocals? Reciprocals are pairs of numbers that when multiplied together, give a product of 1. For example, 5 and $\frac{1}{5}$ are reciprocals, since $5 \times \frac{1}{5} = 1$. The number 1 is called the *identity element* for multiplication—when you multiply by 1, the value of the expression stays the same. This same pattern holds true for fractions and integers—in fact, for all real numbers. Here are a few examples:

$$\frac{1}{2} \div \frac{1}{4} = 2 \qquad {}^{-}18 \div 6 = {}^{-}3$$
$$\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \qquad 6 \div ({}^{-}18) = {}^{-}\frac{1}{3}$$
$$2 \times \frac{1}{2} = 1 \qquad {}^{-}3 \times ({}^{-}\tfrac{1}{3}) = 1$$

What happens when the dividend and the divisor are the same number? In this case, for example, $6 \div 6$, the quotient is 1. Reverse the 6s and the quotient stays the same—it's 1 in both instances. Likewise, $1 \times 1 = 1$, so 1 is its own reciprocal.

What happens when at least one of the numbers is 0? Consider the situation in which the dividend is 0 and the divisor is 4 ($0 \div 4 = \square$). If we think of the operation of division as repeated subtraction, we can ask ourselves, "How many groups of four can I subtract from zero?" The answer of 0 makes sense. Using a missing-factor interpretation of division, find the factor that when multiplied by 4 equals 0 ($4 \times \square = 0$). Again, the answer, 0, makes sense. To generalize, if the dividend is 0 and the divisor is another number ($0 \div a$ when $a \neq 0$), then the quotient is 0. Now reverse the dividend and divisor: $4 \div 0 = \square$. How many times can you subtract a group of 0 from 4? Once, twice, an infinite number of times? There isn't a unique answer that makes sense using this interpretation of division. Let's try the missing-factor interpretation: $0 \times \square = 4$. There is no solution, because $0 \times 0 = 0$ and 0 times any other number also equals 0.

For $0 \div 0$, you can subtract 0 from 0 any number of times and still get 0, so again there isn't a unique answer that makes sense using a repeated subtraction interpretation of division. Using the missing-factor interpretation ask, 0 times what number equals 0? Any number works. Therefore, for $0 \div 0$, there is also no unique answer. Because there isn't an answer to expressions that are divided by 0 ($a \div 0$), mathematicians have agreed that division by 0 is "undefined." When you try to divide by 0 on a calculator, the readout displays ERROR or E—the operation is undefined and thus has no solution.

The patterns in this activity are related to the fact that multiplication and division are inverse operations—to undo multiplication we divide and to undo division we multiply. In fact, if we multiply the dividend (the 6 in $6 \div 2$) by the reciprocal of the divisor (the divisor is 2, the reciprocal of 2 is $\frac{1}{2}$), we have an equivalent operation ($6 \div 2 = 3$, $6 \times \frac{1}{2} = 3$). The relationship between reciprocals and undoing multiplication and division is very powerful and is used in algebra. One more point: the reason we refer to operations that are based on multiplication and division only as multiplicative is because all divisions can be rewritten as multiplications times the reciprocal of the divisors. ▲

The Associative Property

The associative properties of addition and multiplication relate to how we group numbers in order to find sums and products. These properties are often used in conjunction with the commutative properties for these operations because we regularly change the order of addends or factors prior to regrouping. Addition and multiplication are often referred to as *binary* operations: we can operate on only two numbers at a time (e.g., $3 + 4$, 7×8). If a computation involves three addends, we first add two of the numbers and then add the third to the previous sum—for example, $(3 + 4) + 5 = 7 + 5 = 12$. The associative properties of addition and multiplication state that the way in which three or more addends or factors are grouped before being added or multiplied does not affect the sum or product.

The associative and commutative properties are used to compute mentally. For example, when adding a list of single-digit numbers, many people group digits together that sum to 10—given, for example, $3 + 6 + 7 + 2 + 4$, they might think $(3 + 7) + (4 + 6) + 2$, or $10 + 10 + 2 = 22$. They use the commutative property of addition to switch the order of some addends and then use the associative property of addition to regroup the numbers to simplify the calculations by forming compatible numbers. Compatible numbers, often referred to as "friendly" numbers, are numbers whose sums and products are easy to calculate mentally. For example, 25 and 4 are compatible because $25 \times 4 = 100$, and 35 and 65 are compatible because $35 + 65 = 100$. In general, numbers that can be combined to form multiples of 10 (e.g., 10, 50, 100, 200, 1,000) are compatible.

Too often, students operate on numbers strictly in the order in which they are encountered. Teachers can help students make sense of the associative properties of addition and multiplication by specifically focusing on grouping—asking them to experiment with adding or multiplying numbers in different orders and reflecting on the results and the relative ease or difficulty of the calculations. For example, students might be asked to rewrite expressions like the ones below to show different groupings of the same numbers and to decide which grouping produces the easiest calculation. Notice that the last expression in each list groups compatible numbers—$(24 + 36)$ and (4×5)—and can be computed quickly without paper and pencil.

Using 17, 24, and 36, form three addition expressions	*Using 4, 5 and 17, form three multiplication expressions*
$(24 + 17) + 36$	$(4 \times 17) \times 5$
$24 + (17 + 36)$	$4 \times (17 \times 5)$
$(24 + 36) + 17$	$(4 \times 5) \times 17$

Through experimentation with specific cases, students will realize that the sums and products are the same regardless of which two numbers are operated on first. They may also observe that some calculations (e.g., addends that sum to 10 or its multiples, multiplication involving a factor that has a zero in the ones place) are easier than others. Once students have started to notice patterns and shortcuts, conduct whole-class discussions about what they observe about the mathematical relationships as a way to help all students understand how they can use these properties when computing.

Many procedures students invent when learning to add and subtract make use of both the commutative and associative properties in conjunction with knowledge of place value. For example, how do you mentally compute 76 + 89? One way is to break the numbers down into tens and ones—70 + 6 + 80 + 9—and then add the tens and the ones separately:

$$\begin{array}{rl} 76 & \\ +\ 89 & \\ \hline 150 & (70 + 80) \\ +\ 15 & (6 + 9) \\ \hline 165 & \end{array}$$

When computing mentally we often work left to right rather than right to left—first adding the tens (150), then the ones (15), and then performing a final calculation to get 165. The order in which we add (i.e., ones then tens or tens then ones) does not matter in terms of the final sum. Many adults learned the rule that with all computations except division you start on the right and are not aware that there are many efficient methods for computing that don't follow this rule.

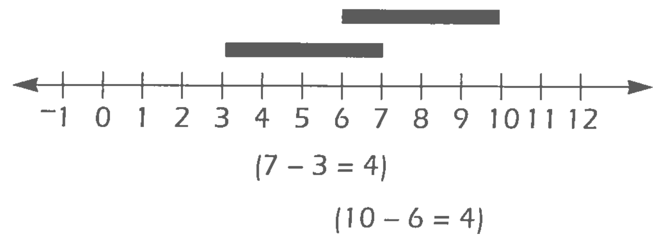
A mental math technique called *compensation* involves reformulating a sum, product, difference, or quotient so that it is easier to work with. For example, rewriting addition computations to form compatible numbers doesn't change the value of the expression. Addends can be decomposed and then recomposed using the commutative and/or associative property to create easier computations. The addition 9 + 5 can be changed to an equivalent addition, 10 + 4, by decomposing 5 into 1 + 4 and adding the 1 and 9. In the computation 76 + 89, the 76 can be decomposed into 75 + 1; using the associative property, the 1 can then be recomposed with the 89 (1 + 89 = 90) to form the equivalent expression of 75 + 90.

For	*Think*	*How it works*
9 + 5	10 + 4	9 + (1 + 4) = (9 + 1) + 4
76 + 89	75 + 90	(75 + 1) + 89 = 75 + (1 + 89)

Subtraction expressions can also be adjusted without changing the final value (difference). But in the case of subtraction, a quantity is added to (or subtracted from) both numbers in the expression.

For	*Think*	*How it works*
13 − 9	14 − 10	(13 + 1) − (9 + 1) = (13 − 9) + (1 − 1)
62 − 28	64 − 30	(62 + 2) − (28 + 2) = (62 − 28) + (2 − 2)

In the examples above, either 1 or 2 is added to both numbers (which does not affect the difference) so that the number subtracted is a friendly, compatible number. The numbers added (in these cases, 1 and 2) are not arbitrary—they were chosen so that both of the subtrahends (9 and 28) would be multiples of 10 ($9 + 1 = 10$ and $28 + 2 = 30$). The reason we can either add or subtract a set quantity to both numbers in a subtraction problem can be illustrated using the concept of the difference as an interval on the number line. We know that $10 - 6 = 4$, because the interval between 10 and 6 is 4. If we move the interval along the number line, say to 7 and 3 by subtracting 3 from both 10 and 6, the interval or difference hasn't changed. Many subtraction problems have a difference of 4 and we can create these problems by moving the interval along the number line, adding or subtracting the same amount each time.



Multiplication and division expressions can also be adjusted using the commutative and associative properties of multiplication as well as the fact that a number can be expressed as a product of its factors. The computation 306×5, for example, can be rewritten as 153×10. Because 306 has 153 and 2 as factors, the expression can be restated as $153 \times 2 \times 5$. Using the associative property of multiplication, the multiplication 2×5 is calculated first, followed by 153×10.

For	*Think*	*How it works*
306×5	153×10	$(153 \times 2) \times 5 = 153 \times (2 \times 5)$

Even though how we group addends or factors does not change the final sums or products, these properties cannot be generalized to number sentences that involve more than one operation. So what happens when we have number sentences with more then one operation? Is grouping important? For example, what is the answer to $2 + 6 \times 3$? Did you get 24 or 20? When there is more than one operation in a computation, how we group the numbers makes a difference in terms of the answer!

Remember that the operations of addition, subtraction, multiplication, and division are binary operations. Namely, we perform these operations on two numbers at a time. So when we consider the order in which to perform computations, we are always working with two numbers and one operation at a time. Rules for the order of operations developed gradually over hundreds of years. Some of the rules evolved naturally, such as the use of parentheses to clarify the intent of the writer of a number sentence, but other rules are somewhat arbitrary and were agreed on by mathematicians as the need for consistency became greater. In particular, with more and more computations being performed by calculators and computers, it became important that everyone agree on a specific order in situations that might be interpreted in many ways. Looking back at $2 + 6 \times 3$, the order of operations indicates that we first multiply 6 times 3 and then add 2 for an answer of 20!

Activity

3

How Do Grouping and Order Affect Answers?

Objective: explore how the order of operations affects answers.

When does it matter which operation is performed first in a problem? Pick two operations to investigate—say addition of 5 and division by 2. Choose four numbers to perform the operations upon (1, 9, 12, and 20) and then switch the operations and record the results in a table. Compare the results. What patterns do you notice?

Starting Number	Add 5, Divide by 2	Divide by 2, Add 5
1		
9		
12		
20		

Try this with the following operations:

- ▲ Addition and subtraction.
- ▲ Multiplication and division.
- ▲ Multiplication and subtraction.

Things to Think About

Examining the two columns that have + 5 followed by ÷ 2 and ÷ 2 followed by + 5, we see that the order of these two operations does matter.

Starting Number	Add 5, Divide by 2	Divide by 2, Add 5
1	3	5.5
9	7	9.5
12	8.5	11
20	12.5	15

The answer is always greater when you divide first and then add. In fact, the difference is exactly 2.5. Why? Let's use x to stand for any number. We can represent "add 5, divide by 2" as $\frac{x+5}{2}$ and "divide by 2, add 5" as $\frac{x}{2} + 5$. Before proceeding, take a minute and decide why the symbolic representations of these two expressions are different. It may help to be explicit about the actions: "add 5 to a number and divide the new sum by 2" compared with "take a number and divide it by 2 and then add 5 to the result." Next represent each expression as the sum of two fractions: $\frac{x+5}{2} = \frac{x}{2} + \frac{5}{2}$ and $\frac{x}{2} + 5 = \frac{x}{2} + \frac{5}{1}$.

Notice that we divide the number, x, by 2 in both expressions ($\frac{x}{2}$), but when we add 5 first, that 5 also gets divided by 2, which gives us 2.5. When we add the 5 after the division, that 5 is unaffected by the division. The difference between 5 and 2.5 is 2.5.

When a computation only has the operations of addition and subtraction, the computations need to be performed in the order that they occur in the expression. Otherwise, we will not consistently get the same answer. For example, in $9 - 3 + 2$, we first subtract 3 from 9 and then add 2 for an answer of 8. If we first added $3 + 2$ before subtracting, a different answer of 4 is calculated.

$(9 - 3) + 2$	$9 - (3 + 2)$
$6 + 2$	$9 - 5$
8	**4**

When we have both the operations of multiplication and division or the operations of multiplication and subtraction, it does matter which operation we perform first.

$30 \div (2 \times 5)$	$(30 \div 2) \times 5$
$30 \div 10$	15×5
3	**75**
$(25 - 12) \times 2$	$25 - (12 \times 2)$
13×2	$25 - 24$
26	**1**

The answers are quite different, depending on which operation is performed first. Explore the relationships between the answers based on which operation is performed first using algebra. Let x represent the starting value. Explain why the answer to $(x - 12) \times 2$ is different from the answer to $x - (12 \times 2)$.

In order to avoid errors, when there are more than two operations in a number sentence, we use the Order of Operations: first do any calculations that are grouped together (often shown with parentheses or the fraction bar), then calculate exponents, then multiply and divide from left to right, and finally add and subtract in order from left to right. Teachers sometimes provide students with the mnemonic PEMDAS (parentheses, exponents, multiplication, division, addition, subtraction) as a way to remember the order of operations. However, many educators believe this mnemonic causes more harm than good. If students have not had opportunities to explore what happens when the order of operations is reversed, they may apply the rules blindly or misinterpret what the letters in PEMDAS represent. For example, instead of performing the operations of multiplication and division in the order they occur from left to right, students sometimes first do all the multiplications followed by all divisions, regardless of order. ▲

The Distributive Property

The distributive property of multiplication over addition allows us to "distribute" a factor, a, to two different addends (or in more math terms, "over" two different addends), b and c: $a(b + c) = ab + ac$. It is used extensively when computing mentally. For example, how would you mentally multiply 3 times 58? One approach is to think of the 58 as 50 + 8 and use the distributive property of multiplication over addition:

$$3 \times 58 \Rightarrow 3 \times (50 + 8) \Rightarrow (3 \times 50) + (3 \times 8) \Rightarrow 150 + 24 \Rightarrow 174$$

The distributive property is often used in connection with compensation; that is, to calculate 3×58, we adjust it to make an easier calculation. In the example above, we can round 58 to 60 and then subtract 2. This quantity is then multiplied by 3 $(3 \times (60 - 2))$.

But can the distributive property of multiplication be applied to subtraction? Yes, since we can decompose a number into the sum of two addends and the addends can be negative or positive. For example, think of the subtraction $60 - 2$ as equivalent to $60 + {}^{-}2$, since adding ${}^{-}2$ is equivalent to subtracting 2. Thus, we can distribute multiplication over the $(60 - 2)$, which is equivalent to distributing multiplication over $(60 + {}^{-}2)$:

$$3 \times 58 \Rightarrow 3 \times (60 - 2) \Rightarrow (3 \times 60) - (3 \times 2) \Rightarrow 180 - 6 \Rightarrow 174$$
$$3 \times 58 \Rightarrow 3 \times (60 + {}^{-}2) \Rightarrow (3 \times 60) + (3 \times {}^{-}2) \Rightarrow 180 + {}^{-}6 \Rightarrow 174$$

The distributive property may also be applied to division expressions. Consider the expression $132 \div 12$. An equivalent expression using multiplication is $132 \times \frac{1}{12}$. If we rewrite 132 as $(120 + 12)$, we can distribute the multiplication by $\frac{1}{12}$ over both addends.

$$132 \times \tfrac{1}{12} \Rightarrow (120 + 12) \times \tfrac{1}{12} \Rightarrow (120 \times \tfrac{1}{12}) + (12 \times \tfrac{1}{12}) \Rightarrow 10 + 1 \Rightarrow 11$$

Because we can rewrite all division (except by 0) as multiplication, we can also apply the distributive property of multiplication over addition to the problem:

$$132 \div 12 \Rightarrow (120 + 12) \div 12 \Rightarrow (120 \div 12) + (12 \div 12) \Rightarrow 10 + 1 \Rightarrow 11$$

Not only is the distributive property useful when computing mentally, but it also is applied in our standard multiplication and division algorithms. When students study algebra, they learn how to apply the distributive property when multiplying polynomials.

Properties of whole numbers are used extensively when computing. The particular numbers involved in a calculation determine when it makes sense to use the commutative, associative, or distributive property or some combination of them. Although it's not important for young students to be able to identify these properties by name, it is extremely important that teachers understand how these properties enable students to compute accurately. Thus, teachers can highlight the essential components of students' solution methods (e.g., multiplying by one, changing the order, grouping different numbers together, undoing an operation, or decomposing and recomposing) in order to help students analyze why procedures work to produce correct answers. In middle school, students revisit these properties in preparation for algebra. If they have a solid grasp of when and how these properties are used in arithmetic, they will be better able to generalize the relationships and represent them with variables.

2. Basic Facts

The basic facts in addition and multiplication are all possible sums and products of the digits 0 through 9. Although technically there are 100 addition facts and 100 multiplication facts, because of the commutative property there are actually only 55 unique basic facts for each operation:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

addition basic facts

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

multiplication basic facts

Furthermore, adding 0 or multiplying by 1 poses no problems for memorization because of the identity properties of addition and multiplication ($a + 0 = a$ and $a \times 1 = a$). Thus, there are even fewer than 55 facts to learn.

Since knowledge of the addition facts can be applied to subtraction and knowledge of multiplication facts can be applied to division, there is no need to learn subtraction and division facts as isolated procedures. However, this assumes that the relationships between inverse operations (operations that "undo" each other—addition and subtraction, and multiplication and division) have been thoroughly explored, discussed, and internalized. When students understand the inverse relationship between operations, you can encourage them to use what they know about multiplication, for example, to learn about division. If they need to solve $36 \div 9$, suggest that they think, 9 *times what number is equal to* 36? After generating equations that use the numbers 36, 9, and 4, they can then discuss how and why these multiplication and division equations are related. You might also ask them to write two stories (one multiplication, one division) that use the numbers 36, 9, and 4 and then to reflect on how the stories are similar and different. You can also introduce activities that feature division "near facts." For example, if we convert $50 \div 8$ to a multiplication problem, 8×7 is too large, 8×5 is too small, and 8×6 is still too small but closer. Using this information, students can then determine that $50 \div 8 = 6$ with a remainder of 2.

Students need to know the basic facts in order to be both efficient and accurate, whether their calculations are performed mentally or by applying paper-and-pencil algorithms. Instruction involving basic facts should focus on making sense, highlight strategies for remembering facts, and be connected to all other work with number.

This instruction might include activities that ask students to explore relationships between operations (e.g., addition and subtraction) or to explore properties and then apply the properties to learning other facts. For example, students might determine 8×7 by decomposing 7 into $3 + 4$ and using their knowledge of 8×3 and 8×4 $(24 + 32 = 56)$. This strategy works because of the distributive property $(8 \times (3 + 4))$. Likewise, students can further their knowledge by writing word problems that use the facts in a meaningful context, using concrete objects to connect these situations with symbols, and discussing relationships within fact families.

Learning facts is a gradual process that for most students takes a number of years. (See Chapter 3 for specifics on how skills develop in addition and subtraction.) Eventually, however, students need to commit the basic facts to memory. Naturally, the exact age when a particular student masters these facts varies. In general, however, most students have mastered addition/subtraction facts by the end of third grade and multiplication/division facts by the end of fifth grade. "Mastery" does not imply that students are human calculators able to perform at lightning speed. It means that they know the facts well enough to be efficient and accurate in other calculations.

3. Algorithms

Algorithms, as the term is applied to the arithmetic procedures students traditionally have learned in school, are systematic, step-by-step procedures used to find the solution to a computation accurately, reliably, and quickly. Algorithms, whether performed mentally or with paper and pencil, a calculator, or a computer, are used when an exact answer is required, when an estimate won't suffice. Because they are generalizations that enable us to solve classes of problems, they are very powerful: we can solve many similar tasks (1,345,678 − 987,654 and 134 − 98, for example) using one process. In the best of circumstances, algorithms free up some of our mental capacity so that we can focus on interpreting and understanding a solution in the context of a problem. In the worst of circumstances, algorithms are used when a task could be done mentally or are applied by rote with little understanding of the bigger mathematical picture—why the calculation is important and how the answer will be used.

There are many different algorithms for performing operations with numbers. Some of these algorithms are now referred to as *standard* or *conventional* simply because they have been taught in the majority of U.S. classrooms over the past fifty years. For example, you may have learned (or taught) the standard addition algorithm shown below, in which you "carry" from the ones column to the tens column to the hundreds column:

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 456 \\ +899 \\ \hline 1355 \end{array}$$

Interestingly, some of the conventional algorithms in the United States are not the standard algorithms in Europe or South America. Children around the world learn different computational procedures in school.

Other algorithms are known as *alternative* algorithms—they differ from the standard algorithms for adding, subtracting, multiplying, and dividing. Alternative algorithms also are accurate, reliable, and fast. Alternative algorithms such as the lattice method for multiplication have sometimes been used in schools as enrichment activities. Today many alternative algorithms are part of the elementary mathematics curriculum. Making sense of algorithms can be instructive; students figure out why certain procedures work, which leads them to insights into important ideas such as place value and the distributive property of multiplication over addition.

In the 1990s many researchers and mathematics educators began to question the wisdom of the rote teaching of conventional algorithms to students in the elementary grades. Research has shown that when children simply memorize the steps to complete the standard addition and subtraction algorithms, they lose conceptual understanding of place value. In contrast, students who invent their own procedures or algorithms for solving addition and subtraction problems have a much better understanding of place value and produce more accurate solutions (Kamii 1994; Kamii and Dominick 1998; Narode, Board, and Davenport 1993). Many mathematics educators now suggest that instead of teaching students standard algorithms as the only or best ways to compute with paper and pencil, we provide many opportunities for students to develop, use, and discuss a variety of methods. Having students invent algorithms leads to enhanced number and operation sense as well as flexible thinking (Burns 1994b; Carroll and Porter 1997, 1998).

Many elementary curriculums are designed so that children initially use logical reasoning and their understanding of number (e.g., that 36 can be decomposed into 30 + 6), place value, and mathematical properties to invent their own algorithms and procedures to solve addition and subtraction problems. The purpose of this type of instruction is to extend and expand students' understanding of number, place value, decomposition, and recomposition (see Chapter 1 for elaboration) as they learn to compute. Student-invented or student-generated procedures sometimes are algorithms; that is, they can be generalized to classes of problems and they enable the student to produce accurate answers. (They many not be efficient or easy to use, however.) Other procedures are not algorithms; they may enable a child to calculate a correct answer, but they cannot be generalized to other problems.

However, as students progress through the grades, they need to acquire efficient ways to compute, and student-generated methods may not suffice. Thus, after students have experimented with developing their own methods, teachers often introduce standard and alternative algorithms as a focus of study. When these algorithms are examined and analyzed, not taught in a rote way, students have the opportunity to build on their already established understanding of number and place value to expand their repertoire of efficient, reliable, and generalizable methods.

In schools we often associate the study of algorithms with paper-and-pencil procedures. One useful by-product of paper-and-pencil algorithms is that they provide a written record of the processes used to solve a problem. Students can use this record to refine procedures, share what has been accomplished, and reflect on solutions. Keeping a record of the steps in an algorithm is especially important when students are trying to make sense of the reasoning involved in the computation. The activities in this section examine a variety of algorithms and student-invented procedures for whole number computations. The goal is for you to understand why these methods work and to consider the mathematics that students have made sense of in order to use them.

Activity

4

Analyzing Students' Thinking, Addition

Objective: learn some common addition strategies.

Examine the following examples of students' procedures for solving addition problems. First, explain what the student did to obtain a correct answer. Then use the student's algorithm to solve the problem 1367 + 498.

Kelly	*Rudy*	*Andy*		
567	567	567	⇒	600
+ 259	+ 259	+ 259		+ 226
700				$\boxed{826}$
110	200 – 567, 667, 767			
16	50 – 777, 787, 797, 807, 817	259		
$\boxed{826}$	9 – 818, 819, 820, 821, 822	– 33		
	823, 824, 825, $\boxed{826}$	226		

Things to Think About

Kelly's algorithm is sometimes called the partial sums method. She added the digits in the problem by place value, starting with the largest place value (hundreds)—500 + 200 = 700, 60 + 50 = 110, 7 + 9 = 16. After calculating the partial sums, Kelly added them (700 + 110 + 16 = 826). Kelly is able to decompose numbers into hundreds, tens, and ones, add like units, and then recompose the three subtotals to produce the final sum. Since her use of this algorithm implies that she understands place value through hundreds, she would probably be able to generalize this approach to four-digit and larger sums. To solve 1367 + 498 using Kelly's method you would figure this way:

$$\begin{array}{r} 1367 \\ +\ 498 \\ \hline 1000 \\ 700 \\ 150 \\ 15 \\ \hline 1865 \end{array}$$

How did Rudy solve 567 + 259? It appears that he started with 567 and either counted on or added on, first by hundreds, then by tens, finally by ones. He started with the hundreds, writing 567, 667, 767. Then he continued with five tens: 777, 787, 797, 807, 817. Finally he finished with the ones: 818, 819, 820, 821, 822, 823, 824, 825, 826. While this procedure works, it is prone to error, especially when an increase in one grouping necessitates an increase in the next one up (797 to 807, for example). To solve 1367 + 498 using Rudy's method, round 498 up to 500, count on by hundreds (1467, 1567, 1667, 1767, 1867), and then count backward by ones (1866, 1865). Sometimes students use an invented procedure for a short period of time and then move on to another, more efficient procedure or algorithm. The importance of classroom discussion about solution procedures cannot be overemphasized—students often learn about other approaches when their classmates describe their method.

Andy used an approach different from the other two students in that he did not decompose the numbers based on hundreds, tens, and ones. Instead, he changed both numbers to other numbers that he thought would be easier to use. He started by adding 33 to 567 to get 600. He then subtracted the 33 from 259 to get 226. Finally, he added his two adjusted numbers: 600 + 226 = 826. Andy's method works because of the associative and commutative properties. The identity property of addition also comes into play: adding zero—33 + (−33) = 0—doesn't change the sum:

$$\begin{aligned} 567 + 259 &= (567 + 259) + (33 + {}^{-}33) \\ &= (567 + 33) + (259 + {}^{-}33) \\ &= (567 + 33) + (259 - 33) \\ &= 600 + 226 \\ &= 826 \end{aligned}$$

Andy's procedure is very efficient with an addition such as 1367 + 498, because it's easy to "see" how to change 498 to 500 and compensate by subtracting two from 1367. However, this approach may not be all that ideal with problems such as 2418 + 1725 (though it will work), since the additions and subtractions leading to the adjusted numbers may require a lot of mental energy. ▲

Activity

5 Analyzing Students' Thinking, Subtraction

Objective: learn some common subtraction strategies.

Examine the following examples of students' procedures for solving the same subtraction problem. What did each student do to obtain a correct answer? Why does the student's algorithm work?

Caitlin

$$\begin{array}{rl} 63 & \\ -\ 18 & \quad 10 + 8 \\ \hline \multicolumn{2}{l}{63 - 10 = 53} \\ \hline \multicolumn{2}{l}{52, 51, 50, 49,} \\ \multicolumn{2}{l}{48, 47, 46, 45} \\ \boxed{45} & \end{array}$$

Louis

$$\begin{array}{r} {}^{13} \\ 6\cancel{3} \\ -{}^{2}\cancel{1}8 \\ \hline \boxed{45} \end{array}$$

Kenley

$$\begin{array}{r} 63 \\ -18 \\ \hline 28 \\ 38 \\ 48 \\ 58 \\ +\ 5 \\ \hline \end{array} \quad \left.\begin{array}{r} 10 \\ 10 \\ 10 \\ 10 \end{array}\right\} \quad \text{or} \quad \begin{array}{r} 40 \\ 5 \\ \hline \boxed{45} \end{array}$$

Things to Think About

Caitlin solved this computation by inventing a procedure based on her understanding of place value and counting back. First she decomposed 18 into 10 + 8, then she removed the 10 from 63 (63 − 10 = 53), and finally she counted back 8 from 53 to 45. This procedure is commonly seen in second-grade classrooms in which children have been encouraged to invent their own procedures that make sense to them. Let's try Caitlin's procedure with 152 − 39.

$$\begin{array}{rl} 152 & \\ -\ 39 & \quad 30 + 9 \\ \hline 100 & \\ 52 & - 30 = 22 \\ 122 & \\ \hline \end{array}$$

122, 121, 120, 119, 118, 117, 116, 115, 114, $\boxed{113}$

While it is unlikely that you would use this procedure for larger multidigit subtraction problems, adults do use a combination of methods including counting back with simple calculations such as determining elapsed time (e.g., *for how many hours do I pay the baby-sitter if it is now 12:45 A.M. and I left at 4:30 P.M.?*). Depending on the context and the numbers in a problem, simple counting procedures can be quick and efficient.

Louis moved to the United States when he was eleven and learned this algorithm in school in Italy. What are the steps in Louis's algorithm? First, Louis noticed that there were not enough ones in the ones place (3) to subtract 8, so he changed the 3 to a 13. Because he added 10 to the 63 (63 + 10 = 60 + 13), he had to add 10 to the 18 (10 + 18 = 28) in order not to change the problem. Louis recorded the 10 added to the 18 by changing the 1 in the tens place to a 2. He then subtracted the ones (13 − 8 = 5) and the tens (6 − 2 = 4) to obtain the answer of 45. This algorithm uses compensation (see page 37 in this chapter)—if you add (or subtract) the same number to both the minuend and the subtrahend in a subtraction problem, the difference is not affected.

$$\begin{array}{r} 63 + 10 \\ -\ 18 + 10 \\ \hline 45 \end{array} \Rightarrow \begin{array}{r} 73 \\ -\ 28 \\ \hline 45 \end{array} \qquad \begin{array}{r} 63 + 10 = \\ -18 + 10 = \\ \hline \end{array} \begin{array}{r} 60 + 13 \\ 20 + \ 8 \\ \hline 40 + \ 5 \end{array}$$

What is interesting about Louis's algorithm is that the 10 is added to different place values, which simplifies the computation. Let's try his procedure with 81 − 58:

$$\begin{array}{r} 8\overset{11}{\cancel{1}} \\ -\ {}^{6}\cancel{5}8 \\ \hline 23 \end{array}$$

Kenley understands the inverse relationship between addition and subtraction and used an approach sometimes referred to as "adding on" or "counting up." It is often used by students for subtraction word problems involving money when making change. For example, to solve the problem, "How much change should you receive from a $5.88 purchase if you give the clerk a $10 bill?," you can add on from $5.88 to $10.00 ($.12 makes $6.00 plus $4.00 makes $10.00, the change is $4.12). The amount Kenley added on to 18 to reach 63 was her answer. She counted by tens until she was close to but not above 63: 28, 38, 48, 58. She recorded that she had added four tens, or 40. When she realized that five more ones would bring her to 63 (58 + 5 = 63), she had her final answer of 45. Let's use Kenley's method to solve 81 − 58:

$$\begin{array}{r} 81 \\ -\ 58 \\ \hline \end{array} \qquad \begin{array}{rcc} 58 & 68,\ 78 & \\ +\boxed{20} & \Leftarrow & \\ \hline 78 & 79,\ 80,\ 81 & \boxed{23} \\ +\ \boxed{3} & \Leftarrow & \\ \hline 81 & & \end{array}$$

The standard subtraction algorithm is the one that most adults were taught when students. It is an efficient method that is based on decomposition and regrouping. Let's examine how Juan solves 152 − 39.

$$\begin{array}{r} 1\overset{4}{\cancel{5}}\overset{12}{\cancel{2}} \\ -\ 39 \\ \hline 113 \end{array}$$

Juan's actions are based on the fact that numbers can be decomposed into hundreds, tens, and ones ($152 = 100 + 50 + 2$) and recomposed in less efficient groupings ($152 = 100 + 40 + 12$) that can be used in certain types of problems. He notes that since there are only 2 ones in the ones place and he wants to subtract 9 ones, he needs to regroup 1 ten from the tens place and combine these 10 ones with the 2 ones for a total of 12 ones. This leaves only 4 tens in the tens place. Now he can subtract by place value ($12 - 9 = 3$, $40 - 30 = 10$, $100 - 0 = 100$). In the most abbreviated version of this algorithm, Juan subtracts $4 - 3$ and $1 - 0$ as if these numbers represented ones, but knows that by their placement in the tens and hundreds columns, respectfully, they represent $40 - 30$ and $100 - 0$. As with many algorithms, this one can be applied by rote. If students solve problems such as this one using a memorized procedure, they usually are unable to explain their actions. For example, they can't tell you what the 4 or the 12 above the tens and ones places represent and why those numbers are being used. Students who are unclear on the purpose of regrouping sometimes record answers that make no sense. Or, if students report that "this is how you do it" when asked to explain their steps, it is a signal that they may have memorized the procedures to execute the algorithm without understanding them. However, if students understand the relationships involved, they will have a solid grasp of place value and decomposition/recomposition and will find this algorithm very useful.

The application of the standard algorithm is more difficult when the minuend contains one or more zeroes. Teachers and students alike agree that in these situations the standard algorithm is confusing and hard to use! The regrouping process involves many steps and requires that students understand the relationships between larger place values. Let's take a look at $5009 - 836$.

$$\begin{array}{r} 9 \\ 4\,\cancel{10}\,10 \\ \cancel{5}\cancel{0}\cancel{0}9 \\ -\ 836 \\ \hline 4173 \end{array}$$

When we decompose the number 5009 into place values, there are no tens or hundreds to regroup in different ways. This means that we must decompose the 5000 into $4000 + 1000$ and regroup the 1000 as 10 hundreds. This is shown in the algorithm by crossing out the 5 (5 thousand), replacing it with a 4 (4 thousand) and putting a 10 over the hundreds to indicate the regrouping of 1000 as 10 hundreds. Then we decompose the 10 hundreds into $900 + 100$ and regroup the 100 as 10 tens. Using symbols, this is recorded by crossing out the 10 above the hundreds place and writing a 9 (9 hundred) and placing a 10 above the tens place (10 tens). Finally, we are ready to subtract place by place ($9 - 6$ ones, $10 - 3$ tens, $9 - 8$ hundreds, and $4 - 0$ thousands).

Whereas the standard algorithm can be quite efficient, some students find it easier to use compensation when subtraction involves numerous zeroes in the minuend. For example, subtract 9 from both numbers ($5009 - 9$ and $836 - 9$). The equivalent subtraction problem of $5000 - 827$ can be solved in a variety of ways. Using Kenley's "adding on" method, students might count on from 827 starting at the ones place ($827 \Rightarrow 3 + 70 + 100 + 4000$). Or they might add 173 to both numbers ($5000 + 173 = 5173$, $837 + 173 = 1000$) and perform the simpler calculation, $5173 - 1000 = 4173$. Take a minute and think about what students need to understand in order to be able to use a variety of solution methods. ▲

6

Analyzing Students' Thinking, Multiplication

Objective: learn some common multiplication strategies.

Examine the following examples of students' procedures for solving the same two multiplication problems. What did each student do to obtain a correct answer? Why does the student's algorithm work?

Sasha

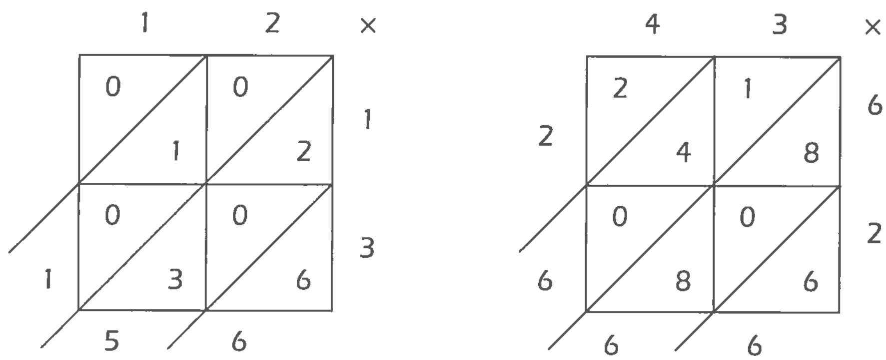
$$\begin{array}{r} 12 \\ \underline{\times 13} \\ 6 \\ 30 \\ 20 \\ \underline{+100} \\ 156 \end{array} \qquad \begin{array}{r} 43 \\ \underline{\times 62} \\ 6 \\ 80 \\ 180 \\ \underline{+2400} \\ 2666 \end{array}$$

Emily

$$\begin{array}{rr} 12 \times 13 & \\ 10 \times 13 = & 130 \\ 2 \times 13 = & \underline{26} \\ & 156 \end{array}$$

$$\begin{array}{rr} 43 \times 62 & \\ 20 \times 62 = & 1240 \\ 20 \times 62 = & 1240 \\ 3 \times 62 = & \underline{186} \\ & 2666 \end{array}$$

Tabitha

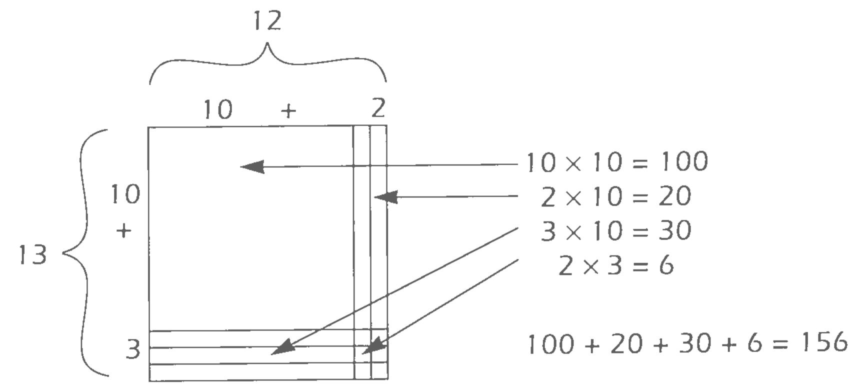


$12 \times 13 = 156$ $\qquad$ $43 \times 62 = 2666$

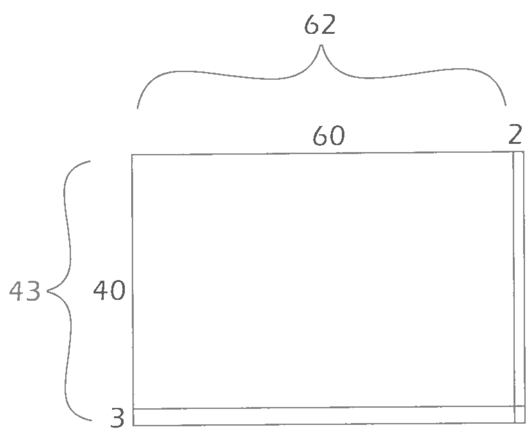
Things to Think About

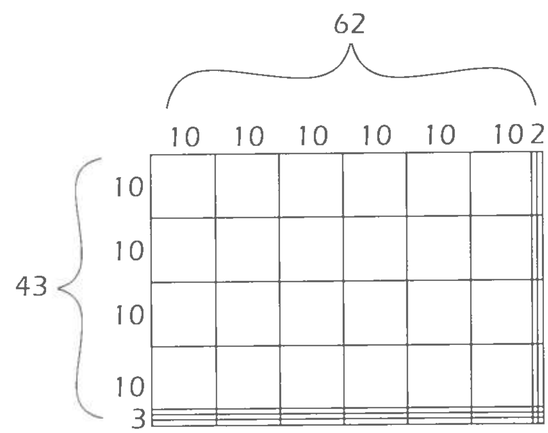
Sasha's algorithm, the partial product algorithm, is often taught to help students understand the steps in a multiplication problem prior to learning the standard multiplication algorithm. To solve 12×13, Sasha decomposed both numbers into tens and ones—$(10 + 2) \times (10 + 3)$—and completed four multiplications (3×2, 3×10, 2×10, and 10×10) to obtain partial products. He then added these partial products together to obtain the answer. The partial product algorithm works because of the distributive property: 12×13 is equivalent to $12 \times (10 + 3)$, which is also equivalent to $(10 + 2) \times (10 + 3)$. Students revisit this application of the distributive property in algebra (where it is known as the "foil method") when they multiply polynomials such as $(x + 3)(x + 2)$. The partial product algorithm is sometimes presented using a rectangular array to help students visualize multiplication in terms of the area of a region. A rectangle with the length and width of the factors in the multiplication problem (12 by 13, in this case) is first constructed on graph paper. Each side is partitioned into tens and ones. Lines are drawn to show the different regions, and the area of each region is determined.

These areas are then added together. (Base ten blocks—hundreds, tens, and ones—can be used to fill in the regions to represent the areas concretely.)



Try the rectangular array approach with Sasha's second problem, 43×62. One side of the 43 by 62 rectangle can be marked to show $40 + 3$, or $10 + 10 + 10 + 10 + 3$. The other side can be marked to show $60 + 2$, or $10 + 10 + 10 + 10 + 10 + 10 + 2$.





Emily broke down 12×13 into computations simple enough to do in her head. She used a method similar to the partial product algorithm in that she found some partial products and then added. Her method also relies on the distributive property, distributing 13 across the 10 and the 2: $(13 \times 10) + (13 \times 2)$.

$12 \times 13 \quad \Rightarrow \quad 10 \times 13 = 130 \quad \Rightarrow \quad 2 \times 13 = 26 \quad \Rightarrow \quad 130 + 26 = 156$

Emily's solution to 43×62 again uses the distributive property, but in order to keep the calculations easy enough to do in her head she first thought of 43 as $(20 + 20 + 3)$ and then calculated the partial products of each of these multiplied by 62. Since there were many steps in this calculation, she jotted them down:

$$20 \times 62 = 1240$$
$$20 \times 62 = 1240$$
$$3 \times 62 = 186$$
$$1240 + 1240 + 186 = 2666$$

Notice that the individual procedures produce individual partial products, which are then added to find the final product. Clearly, Emily understands what happens when one multiplies and is ready to learn a quicker, more efficient algorithm.

Tabitha used the *lattice* method of multiplication. It is an alternative algorithm students almost never invent on their own; it is taught because analyzing it can help students make sense of the role of regrouping in multidigit multiplication. To multiply 43×62, Tabitha wrote the two factors, 43 and 62, above and to the right of the lattice and recorded partial products in cells with the tens value above the diagonal line and the ones value below the diagonal line (see 3×6 and 3×2 in the figure on page 49). After she had done this for 4×6 and 4×2 as well, she extended the diagonal lines and added the numbers in each diagonal. If the sum in a diagonal was greater than 9 (such as the second diagonal from the right), she regrouped the 10 tens as 1 hundred into the next diagonal to the left. The final product is read starting from the left side of the lattice. The product of 43×62 is 2666.

Why does the lattice method of multiplication work? While it looks very different, this algorithm is similar to the standard multiplication algorithm. The number 43 is multiplied by 2 (the bottom row) and by 60 (the top row) for a total of 62 times. While it appears that you are only multiplying by 6, notice how the numbers in the cells in the top row are all shifted over one place because of the diagonals. This has the effect of placing the numbers in positions that represent multiplication by 60, not 6. In the standard multiplication algorithm we also shift the placement of digits (often using a zero as a place holder) in order to represent multiplying by 60:

$$\begin{array}{r l} 43 & \\ \times\ 62 & \\ \hline 86 & \\ 2580 & \Leftarrow \text{shift over or place a 0 in the ones place} \\ \hline 2666 & \end{array}$$

One feature of the lattice algorithm that makes it especially appealing to some students is that multiplication and addition within the algorithm are kept separate. Basic facts are used to fill the cells but adding only occurs when determining the sums of the diagonals. ▲

Activity

Analyzing Students' Thinking, Division

Objective: learn some common division strategies.

Examine the following examples of students' procedures for solving division problems. What did each student do to obtain a correct answer? Why does the student's algorithm work?

Doug

$$\begin{array}{r|r} 137 \text{ r } 4 & \\ 5\overline{)689} & \\ -\ 500 & 100 \bullet 5 \\ \hline 189 & \\ 50 & 10 \bullet 5 \\ \hline 139 & \\ 50 & 10 \bullet 5 \\ \hline 89 & \\ 50 & 10 \bullet 5 \\ \hline 39 & \\ 35 & 7 \bullet 5 \\ \hline 4 & 137 \bullet 5 \end{array}$$

Nancy

$$\begin{array}{r} 137 \text{ r } 4 \\ 5\overline{)689} \\ \underline{5} \\ 18 \\ \underline{15} \\ 39 \\ \underline{35} \\ 4 \end{array}$$

Madelaine

$$\begin{array}{r c r} 500 \div 5 & = & 100 \\ 100 \div 5 & = & 20 \\ 50 \div 5 & = & 10 \\ 30 \div 5 & = & 6 \\ 9 \div 5 & = & 1 \text{ r } 4 \\ \hline 689 \div 5 & = & 137 \text{ r } 4 \end{array}$$

Things to Think About

Doug's method is sometimes referred to as the scaffold algorithm, which uses repeated subtraction to find quotients. For example, in 689 ÷ 5, multiples of 5 are subtracted successively until a remainder less than 5 is obtained. Doug was able to calculate the products of 5 times 100 and 5 times 10 mentally. He then subtracted these amounts from the quotient. Doug was unclear on how many groups of 5 he could subtract at a time so he repeatedly subtracted 10 groups of 5, or 50. Another student solving this problem using the same algorithm might estimate first and subtract out the 30 groups of 5 at one time:

$$\begin{array}{r|r} 5\overline{)689} & \\ \underline{500} & 100 \cdot 5 \\ 189 & \\ \underline{150} & 30 \cdot 5 \\ 39 & \\ \underline{35} & \underline{7 \cdot 5} \\ & 137 \cdot 5 \end{array}$$

While this algorithm can be somewhat slow and cumbersome, accuracy is fairly high. Furthermore, this method is faster if an individual uses estimation to remove multiples of the divisor. (The algorithm can help students improve their estimation skills, since a teacher can highlight the relationship between the divisor, partial quotient, and subtracted amount.)

Nancy used the standard long division algorithm to find the quotient. She has learned the four steps in this algorithm: estimate, multiply, subtract, and bring down. But what is happening in each of these steps, and why does the standard algorithm work? When Nancy estimated the number of groups of 5 she could remove from 6 (1), she actually estimated the number of 5s she could remove from 600 (100) but suppressed the zeroes in the quotient and in the dividend. She then multiplied the 5 and the 1 to obtain the amount she would subtract. Since she had suppressed the zeroes, she simply brought down the next digit (the 8) as if she had subtracted zero:

$$\begin{array}{r} 1 \\ 5\overline{)689} \\ \underline{5} \\ 18 \end{array} \quad \Rightarrow \quad \begin{array}{r} 100 \\ 5\overline{)689} \\ \underline{500} \\ 189 \end{array}$$

In the next step, in which she estimated the number of 5s in 18 (3), she really estimated the number of 5s in 189 (30), but she again suppressed the zero and ignored the 9 (ones):

$$\begin{array}{r} 13 \\ 5\overline{)689} \\ \underline{5} \\ 18 \\ \underline{15} \\ 3 \end{array} \quad \Rightarrow \quad \begin{array}{r} 130 \\ 5\overline{)689} \\ \underline{500} \\ 189 \\ \underline{150} \\ 39 \end{array}$$

When she finally brought down the 9 and estimated the number of 5s in 39, she recorded that seven 5s can be subtracted from 39 with 4 left over:

```
   137 r 4
 5)689
   5
   18
   15
    39
    35
     4
```

After students have made sense of the scaffold algorithm, in which hundreds of 5s and tens of 5s are subtracted, they often have no trouble understanding the steps in the standard long division algorithm. They can explain what is happening and why we can use the shortened version. Students often become much more proficient at long division when they understand why it works.

Madelaine used the distributive property to distribute the division across many subtraction substeps. She subtracted numbers from 689 that were divisible by 5 and recorded the results in a linear fashion. For example, she first subtracted 500 from 689 since 500 is divisible by 5. She then continued by subtracting 100 from 689 (since 500 + 100 = 600) and dividing it by 5. She then subtracted multiples of 10 that were divisible by 5 (50 and then 30), and finally a group of ones. Like Doug, she could have subtracted different amounts that also were divisible by 5 to obtain the correct answer. Madelaine's invented method is based on her understanding that she can decompose 689 and that it is most efficient first to consider the number of hundreds that can be subtracted, then the tens, and finally the ones. Use Madelaine's method to divide 378 by 3.

Division algorithms are not used all that often in today's world. We may use division when we calculate our gasoline mileage or when we eat out with friends and divide the bill evenly. Most of the division we do involves small dividends and single-digit divisors. For more complicated division problems, we reach for the calculator. When using a calculator, however, students should have a ballpark estimate in mind as a check that they have selected keys correctly. And they need to be able to do divisions with paper and pencil. However, practicing long division algorithms with large divisors and even larger dividends has limited value. Much more important is providing students with opportunities to make sense of division situations and procedures. ▲

Teaching Computation

One reason algorithms were invented and codified is so that we can compute accurately and efficiently without having to expend a great deal of mental energy. Yet which algorithm or procedure we use depends on the level of precision needed and whether we are computing mentally or with paper and pencil. (Calculators and computers use programmed algorithms and are by definition quick and accurate.) Students need to understand a variety of approaches to solving problems in order to choose the most appropriate method based on the numbers involved and the complexity of the procedure.

Another factor that affects our choice of algorithms or procedures involves the context of the problem. The context indicates whether we need an accurate answer or if an approximate value will suffice. Sometimes an accurate answer isn't necessary. For example, when checking to see whether we have enough cash for the groceries in our cart we might adjust prices up ($1.89 to $2.00) but not bother to compensate later by subtracting the added amount (11¢) from the final sum. However, when a situation requires a precise answer, we will want to choose a solution method that guarantees accuracy.

Students' facility with the different operations and algorithms develops slowly over time. With experience they become flexible in using a variety of procedures and algorithms based on the numbers in the computations. They also learn to apply their knowledge of contexts to determine when an exact answer is required and when an estimate will suffice. As they progress through the grades, students learn a variety of algorithms for performing computations with rational numbers and integers quickly and easily. Ideally they will understand how the mathematical properties they are familiar with are used with all numbers and how these new algorithms are related to ones they have already encountered.

Questions for Discussion

1. Explain how understanding mathematical properties helps students compute mentally or make sense of operations.
2. A number of commercially available mathematics curriculums are designed so that students invent addition and subtraction algorithms. Why is this considered important?
3. Should students be taught more than one algorithm for an operation? Should all students know the "standard algorithms"? Explain.
4. What might a teacher do when a child explains a procedure for solving a problem but no one in the class seems to understand it, not even the teacher?