Some Data[11]

How often have you caught yourself daydreaming over a doodle of some kind or even over some arithmetic calculation? The following is some very unexpected fall-out based on a "What-If-Not" perspective imposed on just such a situation.

Look at the following number pattern that was arrived at in a spirit of doodling:

$$1 \cdot 3 = 3$$
$$2 \cdot 4 = 8$$
$$3 \cdot 5 = 15$$
$$4 \cdot 6 = 24$$
$$5 \cdot 7 = 35$$

There are many attributes to observe in the above. For example, notice that:

1. In each case there are two factors.
2. The factors in each pair differ by 2.
3. The differences between the products form an interesting pattern:

$$8 - 3 = 5$$
$$15 - 8 = 7$$
$$24 - 15 = 9$$
$$35 - 24 = 11$$

[11] See Stephen I. Brown, "A New Multiplication Algorithm: On the Complexity of Simplicity," *Arithmetic Teacher*, 22(7), 1975, pp. 546–554, and "A Musing on Multiplications," *Mathematics Teaching*, 61, 1974, pp. 26–30.

It appears that the differences form an arithmetic progression; furthermore, the products alternate in parity (odd, even, odd, even, odd). You could take these data and generate many observations, conjectures or questions in the spirit of chapter 3, in which we accept the given.

We could also do a "What-If-Not" on the data in the spirit of chapter 4. With the intention of carrying out such an exploration, let us list one more attribute that was the impetus for this investigation. First look once more at 3, 8, 15, 24, and 35 … as the start of a sequence. If you think in metaphors like "striving," you will be impressed that those numbers in the sequence are all *almost* perfect squares. They all miss by 1. Here is the picture:

$1 \cdot 3 = 3 \rightarrow 4$ (missing by 1)

$2 \cdot 4 = 8 \rightarrow 9$ (missing by 1)

$3 \cdot 5 = 15 \rightarrow 16$ (missing by 1)

$4 \cdot 6 = 24 \rightarrow 25$ (missing by 1)

$5 \cdot 7 = 35 \rightarrow 36$ (missing by 1)

To see where this might lead, let us focus on the attribute that asserts that the factors differ by two. Suppose they are made to differ by four. Then if we still start with 1, we have:

$1 \cdot 5 = 5$

$2 \cdot 6 = 12$

$3 \cdot 7 = 21$

$4 \cdot 8 = 32$

$5 \cdot 9 = 45$

So what? In using the "What-If-Not" strategy we have to ask a question, something we have not done yet. Let us choose as a question something that comes out of the last attribute we observed earlier, that the pattern almost yields squares. Let us ask, "Can we get that again?"

$1 \cdot 5 = 5 = \textcircled{4} + 1$

$2 \cdot 6 = 12 = \textcircled{9} + 3$

$3 \cdot 7 = 21 = \textcircled{16} + 5$

$4 \cdot 8 = 32 = \textcircled{25} + 7$

$5 \cdot 9 = 45 = \textcircled{36} + 9$

Although this pattern yields squares, the correction factors form an arithmetic progression (1, 3, 5, 7, 9). In our original metaphor of "striving," the correction factor for all numbers was the same, namely, the number 1. Can we find something like that here? If we try for 9 rather than 4 as the "striven square" for 1 • 5, let's see what emerges.

$$1 \cdot 5 = 5 = \textcircled{9} - 4 = 3^2 - 2^2$$

$$2 \cdot 6 = 12 = \textcircled{16} - 4 = 4^2 - 2^2$$

$$3 \cdot 7 = 21 = \textcircled{25} - 4 = 5^2 - 2^2$$

$$4 \cdot 8 = ?$$

$$5 \cdot 9 = ?$$

Notice that here we have the same correction factor –4 in every case. Furthermore, that correction factor itself is a perfect square! As we look back at the original data, we realize that there too the correction factor, 1, is also a perfect square.

At this point, you are probably tempted to explore another variation of the product pairs. Again, let us strive for squares given the following factors. (What kind of number is the correction factor itself?

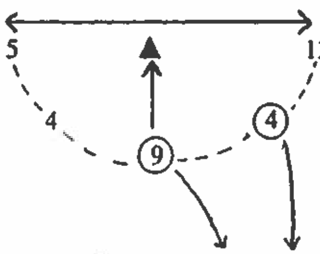
$$1 \cdot 7 = 7 = \textcircled{4^2} - ???$$

$$2 \cdot 8 = 16 = \textcircled{5^2} - ???$$

Finish up on your own!

There is a lot to explore just following this particular line of thought. Can you calculate 5 • 13 so that the "striven number" and the correction factor are both squares? Can you find those squares in an efficient manner?

If you think of the two numbers as sitting on the ends of a seesaw, then it is easy to figure out how to create the two squares. The following picture suggests what is happening. Nine is midway between 5 and 13, thus "balancing" 5 and 13. It is easy to see that the correction factor is 4.



So: $5 \cdot 13 = 9^2 - 4^2$

The implications of this search are extraordinary; they suggest, ultimately, a new algorithm for multiplying *any* two integers. The search was in fact begun by doing a

"What-If-Not" based on free-floating musing as a start. You might want to investigate whether this newly emerging procedure (multiplying any two numbers in terms of the difference of squares) actually becomes complicated or not. It is much more manageable than you would guess initially.

Before leaving this activity, you might also want to do at least one more "What-If-Not" on the data to see if you can find another starting path, based on your own muse. It might be worth saving your future doodles to see in what unexpected directions later "What-If-Nots" might take you.