In: D. Grouws (Ed.). Handbook for Research on Mathematics Teaching and Learning.
New York: Macmillan, 1992

15

LEARNING TO THINK MATHEMATICALLY: PROBLEM SOLVING, METACOGNITION, AND SENSE MAKING IN MATHEMATICS

Alan H. Schoenfeld
THE UNIVERSITY OF CALIFORNIA—BERKELEY

The goals of this chapter are (1) to outline and substantiate a broad conceptualization of what it means to think mathematically, (2) to summarize the literature relevant to understanding mathematical thinking and problem solving, and (3) to point to new directions in research, development, and assessment consonant with an emerging understanding of mathematical thinking and the goals for instruction outlined here.

The use of the phrase "learning to think mathematically" in this chapter's title is deliberately broad. Although the original charter for this chapter was to review the literature on problem solving and metacognition, the literature itself is somewhat ill defined and poorly grounded. As the literature summary will make clear, *problem solving* has been used with multiple meanings that range from "working rote exercises" to "doing mathematics as a professional"; *metacognition* has multiple and almost disjoint meanings (from knowledge about one's thought processes to self-regulation during problem solving) that make it difficult to use as a concept. This chapter outlines the various meanings that have been ascribed to these terms and discusses their role in mathematical thinking. The discussion will not have the character of a classic literature review, which is typically encyclopedic in its references and telegraphic in its discussions of individual papers or results. It will, instead, be selective and illustrative, with main points illustrated by extended discussions of pertinent examples.

Problem solving has, as predicted in the 1980 *Yearbook* of the National Council of Teachers of Mathematics (Krulik, 1980, p. xiv), been the theme of the 1980s. The decade began with NCTM's widely heralded statement, in its *Agenda for Action*, that "problem solving must be the focus of school mathematics" (NCTM, 1980, p. 1). It concluded with the publication of *Everybody Counts* (National Research Council, 1989) and the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), both of which emphasize problem solving. One might infer, then, that there is general acceptance of the idea that the primary goal of mathematics instruction should be to have students become competent problem solvers. Yet, given the multiple interpretations of the term, the goal is hardly clear. Equally unclear is the role that problem solving, once adequately characterized, should play in the larger context of school mathematics. What are the goals for mathematics instruction, and how does problem solving fit within those goals?

Such questions are complex. Goals for mathematics instruction depend on one's conceptualization of what mathematics is, and what it means to understand mathematics. Such conceptualizations vary widely. At one end of the spectrum, mathematical knowledge is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and the relationships among them; knowing mathematics is seen as having mastered these facts and procedures. At the other end of

The author thanks Frank Lester, Indiana University, and Jim Greeno, Stanford University, for their insightful comments on a draft of the manuscript. The current version is much improved for their help. The writing of this chapter was supported in part by the U.S. National Science Foundation through NSF grants MDR-8550332 and BNS-8711342. The Foundation's support does not necessarily imply its endorsement of the ideas or opinions expressed in the paper.

the spectrum, mathematics is conceptualized as the "science of patterns," an (almost) empirical discipline closely akin to the sciences in its emphasis on pattern-seeking on the basis of empirical evidence.

The author's view is that the former perspective trivializes mathematics; that a curriculum based on mastering a corpus of mathematical facts and procedures is severely impoverished—in much the same way that an English curriculum would be considered impoverished if it focused largely, if not exclusively, on issues of grammar. The author characterizes the mathematical enterprise as follows:

> Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making. (Schoenfeld, forthcoming)

This notion of mathematics has gained increasing currency as the mathematical community has grappled, in recent years, with issues of what it means to know mathematics and to be mathematically prepared for an increasingly technological world. The following quotation from *Everybody Counts* typifies the view, echoing themes in the NCTM *Standards* (NCTM, 1989) and *Reshaping School Mathematics* (National Research Council, 1990a).

> Mathematics is a living subject which seeks to understand patterns that permeate both the world around us and the mind within us. Although the language of mathematics is based on rules that must be learned, it is important for motivation that students move beyond rules to be able to express things in the language of mathematics. This transformation suggests changes both in curricular content and instructional style. It involves renewed effort to focus on
>
> - seeking solutions, not just memorizing procedures;
> - exploring patterns, not just memorizing formulas;
> - formulating conjectures, not just doing exercises.
>
> As teaching begins to reflect these emphases, students will have opportunities to study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized. They will be encouraged to see mathematics as a science, not as a canon, and to recognize that mathematics is really about patterns and not merely about numbers. (National Research Council, 1989, p. 84)

From this perspective, learning mathematics is empowering. Mathematically powerful students are quantitatively literate. They are capable of interpreting the vast amounts of quantitative data they encounter on a daily basis and of making balanced judgments on the basis of those interpretations. They use mathematics in practical ways, from simple applications such as using proportional reasoning for recipes or scale models, to complex budget projections, statistical analyses, and computer modeling. They are flexible thinkers with a broad repertoire of techniques and perspectives for dealing with novel problems and situations. They are analytical, both in thinking through issues themselves and in examining the arguments put forth by others.

This chapter is divided into three main parts, the first two of which constitute the bulk of the review. The first part, "Toward an Understanding of 'Mathematical Thinking,' " is largely historical and theoretical, having as its goals the clarification of terms like *problem, problem solving,* and *doing mathematics.* It begins with "Immediate Background: Curricular Trends in the Latter 20th Century," a brief recapitulation of the curricular trends and social imperatives that produced the focus on problem solving as the major goal of mathematics instruction in the 1980s. The next section, "On Problems and Problem Solving: Conflicting Definitions," explores contrasting ways in which the terms *problem* and *problem solving* have been used in the literature and the contradictions that have resulted from the multiple definitions and the epistemological stances underlying them. "Enculturation and Cognition" outlines recent findings suggesting the large role of cultural factors in the development of individual understanding. "Epistemology, Ontology, and Pedagogy Intertwined" describes current explorations into the nature of mathematical thinking and knowing and the implications of these explorations for mathematical instruction. The first part concludes with "Goals for Instruction and a Pedagogical Imperative."

The second part, "A Framework for Understanding Mathematical Cognition," provides more of a classical empirical literature review. "The Framework" briefly describes an overarching structure for the examination of mathematical thinking that has evolved over the past decade. It will be argued that all of these categories—core knowledge, problem-solving strategies, effective use of one's resources, having a mathematical perspective, and engagement in mathematical practices—are fundamental aspects of thinking mathematically. The sections that follow elaborate on empirical research within the categories of the framework. "Resources" describes our current understanding of cognitive structures: the constructive nature of cognition, cognitive architecture, memory, and access to it. "Heuristics" describes the literature on mathematical problem-solving strategies. "Monitoring and Control" describes research related to the aspect of metacognition known as self-regulation. "Beliefs and Affects" considers individuals' relationships to the mathematical situations they find themselves in and the effects of individual perspectives on mathematical behavior and performance. Finally, "Practices" focuses on the practical side of the issue of socialization discussed in the first part, describing instructional attempts to foster mathematical thinking by creating microcosms of mathematical practice.

The third part, "Issues," raises some practical and theoretical points of concern as it looks to the future. It begins with a discussion of issues and terms that need clarification and then points to the need for an understanding of methodological tools for inquiry into problem solving. It continues with a discussion of unresolved issues in each of the categories of the framework discussed in the second part, and concludes with a brief commentary on important issues in program design, implementation, and assessment. The specification of new goals for mathematics instruction consonant with current understandings of what it means to think mathematically carries with it an obligation to specify assessment techniques—methods for determining whether students are achieving those goals. Some preliminary steps in those directions are considered.

TOWARD AN UNDERSTANDING OF "MATHEMATICAL THINKING"

Immediate Background: Curricular Trends in the Latter 20th Century

The American mathematics education enterprise is now undergoing extensive scrutiny, with an eye toward reform. The reasons for the reexamination, and for a major overhaul of the current mathematics instruction system, are many and deep. Among them are the following:

- Poor American showings on international comparisons of student competence. On objective tests of mathematical "basics," students in the United States score consistently near the bottom, often grouped with third-world countries (International Association for the Evaluation of Educational Achievement, 1987; National Commission on Excellence in Education, 1983). Moreover, the mathematics education infrastructure in the United States differs substantially from those of its Asian counterparts, whose students score at the top. Asian students take more mathematics and have to meet much higher standards both at school and at home (Stevenson, Lee, & Stigler, 1986).
- Mathematics dropout rates. From grade 8 on, America loses roughly half of the student pool taking mathematics courses. Of the 3.6 million ninth graders taking mathematics in 1972, for example, fewer than 300,000 survived to take a college freshman mathematics class in 1976; 11,000 earned bachelor's degrees in 1980, 2700 earned master's degrees in 1982, and only 400 earned doctorates in mathematics by 1986. (National Research Council, 1989, 1990a).
- Equity issues. Of those who drop out of mathematics, there is a disproportionately high percentage of women and minorities. The effect, in our increasingly technological society, is that women and minorities are disproportionately blocked access to lucrative and productive careers (National Research Council, 1989, 1990b; National Center of Educational Statistics, 1988a).
- Demographics. "Currently, 8% of the labor force consists of scientists or engineers; the overwhelming majority are White males. But by the end of the century, only 15% of the net new labor force will be White males. Changing demographics have raised the stake for all Americans" (National Research Council, 1989, p. 19). The educational and technological requirements for the work force are increasing, while prospects for more students in mathematics-based areas are not good (National Center of Educational Statistics, 1988b).

The 1980s, of course, is not the first time that the American mathematics enterprise has been declared "in crisis." A major renewal of mathematics and science curricula in the United States was precipitated on October 4, 1957, by the Soviet Union's successful launch of the space satellite *Sputnik*. In response to fears of impending Soviet technological and military supremacy, scientists and mathematicians became heavily involved in the creation of new educational materials, often referred to collectively as the alphabet curricula (SMSG in mathematics, BSCS in biology, PSSC in physics). In mathematics, the "new math" flourished briefly in the 1960s and then came to be perceived as a failure. The general perception was that students had not only failed to master the abstract ideas they were being asked to grapple with in the new math, but they had also failed to master the basic skills that the generations of students who preceded them in the schools had managed to learn successfully. In a dramatic pendulum swing, the new math was replaced by the back-to-basics movement. The idea, simply put, was that the fancy theoretical notions underlying the new math had not worked and that we as a nation should make sure that our students had mastered the basics—the foundation upon which higher-order thinking skills were to rest.

By the end of the 1970s it became clear that the back-to-basics movement was a failure. A decade of curricula that focused on rote mechanical skills produced a generation of students who, for lack of exposure and experience, performed dismally on measures of thinking and problem solving. Even more disturbing, they were no better at the basics than the students who had studied the alphabet curricula. The pendulum began to swing in the opposite direction, toward "problem solving." The first major call in that direction was issued by the National Council of Supervisors of Mathematics in 1977. It was followed by the National Council of Teachers of Mathematics' (1980) *Agenda for Action*, which had as its first recommendation that problem solving be the focus of school mathematics (p. 1). Just as back to basics was declared to be the theme of the 1970s, problem solving was declared to be the theme of the 1980s (see, for example, Krulik, 1980). Here is one simple measure of the turnaround: In the 1978 draft program for the 1980 International Congress on Mathematics Education (ICME IV, Berkeley, California, 1980; see Zweng, Green, Kilpatrick, Pollak, & Suydam, 1983), only one session on problem solving was planned, and it was listed under "unusual aspects of the curriculum." Four years later, problem solving was one of the seven main themes of the next International Congress (ICME V, Adelaide, Australia; see Burkhardt, Groves, Schoenfeld, & Stacey, 1988; Carss, 1986). Similarly, "metacognition," coined in the late 1970s, appeared occasionally in the mathematics education literature of the early 1980s, and then with ever-increasing frequency through the decade. Problem solving and metacognition, the lead terms in this article's title, are perhaps the two most overworked and least understood buzzwords of the 1980s.

This chapter suggests that, on the one hand, much of what passed under the name of problem solving during the 1980s has been superficial, and that were it not for the current "crisis," a reverse pendulum swing might well be on its way. On the other hand, it documents that we now know much more about mathematical thinking, learning, and problem solving than during the immediate post-*Sputnik* years, and that a reconceptualization of both problem solving and mathematics curricula that do justice to it is now possible. Such a reconceptualization will in large part be based in part on advances made in the past decade: detailed understandings of the nature of thinking and learning and of problem-solving strategies and metacognition; evolving conceptions of mathematics as the "science of patterns" and of doing mathematics as an act of sense-making; and cognitive apprenticeship and "cultures of learning."

On Problems and Problem Solving: Conflicting Definitions

In a historical review focusing on the role of problem solving in the mathematics curriculum, Stanic and Kilpatrick (1988) provide the following brief summary:

> Problems have occupied a central place in the school mathematics curriculum since antiquity, but problem solving has not. Only recently have mathematics educators accepted the idea that the development of problem solving ability deserves special attention. With this focus on problem solving has come confusion. The term *problem solving* has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and of why we should teach mathematics in general and problem solving in particular. (p. 1)

Indeed, "problems" and "problem solving" have had multiple and often contradictory meanings through the years—a fact that makes interpretation of the literature difficult. For example, a 1983 survey of college mathematics departments (Schoenfeld, 1983) revealed the following categories of goals for courses that were identified by respondents as "problem solving" courses:

- to train students to "think creatively" and/or "develop their problem solving ability" (usually with a focus on heuristic strategies);
- to prepare students for problem competitions such as the Putnam examinations or national or international Olympiads;
- to provide potential teachers with instruction in a narrow band of heuristic strategies;
- to learn standard techniques in particular domains, most frequently in mathematical modeling;
- to provide a new approach to remedial mathematics (basic skills) or to try to induce "critical thinking" or "analytical reasoning" skills.

The two poles of meaning indicated in the survey are nicely illustrated in two of Webster's (1979, p. 1434) definitions for the term *problem:*

Definition 1: "In mathematics, anything required to be done, or requiring the doing of something."
Definition 2: "A question... that is perplexing or difficult."

Problems as Routine Exercises. Webster's first definition, cited immediately above, captures the sense of the term *problem* as it has traditionally been used, in mathematics instruction. For nearly as long as we have written records of mathematics, sets of mathematics tasks have been with us—as vehicles of instruction, as means of practice, and as yardsticks for the acquisition of mathematical skills. Often such collections of tasks are anything but problems in the sense of the second definition. They are, rather, routine exercises organized to provide practice on a particular mathematical technique that, typically, has just been demonstrated to the student. We begin this section with a detailed examination of such problems, focusing on their nature, the assumptions underlying their structure and presentation, and the consequences of instruction based largely, if not exclusively, in such problem sets. That discussion sets the context for a possible alternative view.

A generic example of a mathematics problem set, with antecedents that Stanic and Kilpatrick (1988) trace to antiquity, is the following excerpt from a late 19th century text, W. J. Milne's *A Mental Arithmetic* (1897). The reader may wish to obtain an answer to problem 52 by virtue of mental arithmetic before reading the solution Milne provides.

> 52. How much will it cost to plow 32 acres of land at \$3.75 per acre?
>
> SOLUTION: \$3.75 is $\frac{3}{8}$ of \$10. At \$10 per acre the plowing would cost \$320, but since \$3.75 is $\frac{3}{8}$ of \$10, it will cost $\frac{3}{8}$ of \$320, which is \$120. Therefore, etc.
>
> 53. How much will 72 sheep cost at \$6.25 per head?
> 54. A baker bought 88 barrels of flour at \$3.75 per barrel. How much did it all cost?
> 55. How much will 18 cords of wood cost at \$6.66 $\frac{2}{3}$ per cord?
>
> [These exercises continue down the page and beyond.]
>
> (Milne, 1897, page 7; cited in Stanic & Kilpatrick, 1988)

The particular technique students are intended to learn from this body of text is illustrated in the solution of problem 52. In all of the exercises, the student is asked to find the product $(A \times B)$, where A is given as a two-digit decimal that corresponds to a price in dollars and cents. The decimal values have been chosen so that a simple ratio is implicit in the decimal form of A. That is, $A = r \times C$, where r is a simple fraction and C is a power of 10. Hence, $(A \times B)$ can be computed as $r \times (C \times B)$. Thus, working from the template provided in the solution to problem 52, the student is expected to solve problem 53 as follows:

$$(6.25 \times 72) = ([\tfrac{5}{8} \times 10] \times 72) = (\tfrac{5}{8} \times [10 \times 72])$$
$$= (\tfrac{5}{8} \times 720) = 5 \times 90 = 450.$$

The student can obtain the solutions to all the problems in this section of the text by applying this algorithm. When the conditions of the problem are changed ever so slightly (in problems 52 to 60 the number C is 10, but in problem 61 it changes from 10 to 100), students are given a suggestion to help extend the procedure they have learned:

> 61. The porter on a sleeping car was paid \$37.50 per month for 16 months. How much did he earn?
>
> SUGGESTION: \$37.50 is $\frac{3}{8}$ of \$100.

Later in this section we will examine, in detail, the assumptions underlying the structure of this problem set and the effects on students of repeated exposure to such problem sets. For now, we simply note the general structure of the section and the basic pedagogical and epistemological assumption underlying its design.

STRUCTURE:

1. A task is used to introduce a technique.
2. The technique is illustrated.
3. More tasks are provided so that the student may practice the illustrated skills.

BASIC ASSUMPTION:

Having worked this cluster of exercises, the students will have a new technique in their mathematical tool kit. Presumably, the sum total of such techniques (the curriculum) reflects the corpus of mathematics the student is expected to master; the set of techniques the student has mastered comprises the student's mathematical knowledge and understanding.

Traditional Uses of "Problem Solving" (in the Sense of Tasks Required To Be Done): Means to a Focused End. In their historical review of problem solving, Stanic and Kilpatrick (1988) identify three main themes regarding its usage. In the first theme, which they call "problem solving as context," problems are employed as vehicles in the service of other curricular goals. They identify five such roles that problems play:

1. *As a justification for teaching mathematics.* "Historically, problem solving has been included in the mathematics curriculum in part because the problems provide justification for teaching mathematics at all. Presumably, at least some problems related in some way to real-world experiences were included in the curriculum to convince students and teachers of the value of mathematics" (p. 13).
2. *To provide specific motivation for subject topics.* Problems are often used to introduce topics with the implicit or explicit understanding that once you have learned the lesson that follows, you will be able to solve problems of this type.
3. *As recreation.* Recreational problems are intended to be motivational, in a broader sense than in number 2, above. They show that "math can be fun" and that there are entertaining uses for the skills students have mastered.
4. *As a means of developing new skills.* Carefully sequenced problems can introduce students to new subject matter and provide a context for discussions of subject-matter techniques.
5. *As practice.* Milne's exercises, and the vast majority of school mathematics tasks, fall into this category. Students are shown a technique and then given problems to practice until they have mastered the technique.

In all five of these roles, problems are seen as rather prosaic entities (recall Webster's first definition) and are used as a means to one of the ends listed above. That is, problem solving is not usually seen as a goal in itself, but solving problems is seen as facilitating the achievement of other goals. Problem solving has a minimal interpretation: working the tasks that have been presented.

The second theme identified by Stanic and Kilpatrick (1988) is "problem solving as skill." This theme has its roots in a reaction to Thorndike's work (Thorndike & Woodworth, 1901). Thorndike's research debunked the simple notion of "mental exercise," in which it was assumed that learning reasoning skills in domains such as mathematics would result in generally improved reasoning performance in other domains. Hence, if mathematical problem solving was to be important, it was not because it made one a better problem solver in general, but because solving mathematical problems was valuable in its own right. This led to the notion of problem solving as skill—a skill still rather narrowly defined (that is, being able to obtain solutions to the problems assigned), but worthy of instruction in its own right. Though there might be some dispute on the matter, this author's perspective is that the vast majority of curricular development and implementation that went on under the name of "problem solving" in the 1980s was of this type.

> Problem solving is often seen as one of a number of skills to be taught in the school curriculum. According to this view, problem solving is not necessarily seen as a unitary skill, but there is a clear skill orientation....
>
> Putting problem solving in a hierarchy of skills to be acquired by students leads to certain consequences for the role of problem solving in the curriculum....[D]istinctions are made between solving routine and nonroutine problems. That is, nonroutine problem solving is characterized as a higher level skill to be acquired after skill at solving routine problems (which, in turn, is to be acquired after students learn basic mathematical concepts and skills). (Stanic & Kilpatrick, 1988, p. 15)

It is important to note that, even though in this second interpretation problem solving is seen as a skill in its own right, the basic underlying pedagogical and epistemological assumptions in this theme are precisely the same as those outlined for Milne's examples in the discussion above. Typically, problem-solving techniques (such as drawing diagrams, looking for patterns when $n = 1, 2, 3, 4, \ldots$) are taught as subject matter, with practice problems assigned so that the techniques can be mastered. After receiving this kind of problem-solving instruction (often a separate part of the curriculum), the students' "mathematical tool kit" is presumed to contain problem-solving skills as well as the facts and procedures they have studied. This expanded body of knowledge presumably comprises the students' mathematical knowledge and understanding.

The third theme identified by Stanic and Kilpatrick (1988) is "problem solving as art." This view, in strong contrast to the previous two, holds that real problem solving (that is, working problems of the "perplexing" kind) is the heart of mathematics, if not mathematics itself. We now turn to that view, as expressed by some notable mathematicians and philosophers.

On Problems That Are Problematic: Mathematicians' Perspectives. As noted earlier, mathematicians are hardly unanimous in their conceptions of problem solving. Courses in problem solving at the university level have goals that range from remediation to critical thinking to developing creativity. Nonetheless, there is a particularly mathematical point of view regarding the role that problems play in the lives of those who do mathematics.

The unifying theme is that the work of mathematicians, on an ongoing basis, is solving problems—problems of the "perplexing or difficult" kind, that is. Halmos makes the claim, quite simply, that solving problems is "the heart of mathematics."

> What does mathematics *really* consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Gödel's proof of undecidability)? Definitions (such as the Menger definition of dimension)? Theories (such as category theory)? Formulas (such as Cauchy's integral formula)? Methods (such as the method of successive approximations)?
>
> Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject, that the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics *really* consists of is problems and solutions. (1980, p. 519)

Some famous mathematical problems are named as such, for example, the four-color problem (which when solved, became the four-color theorem). Others go under the name of hypothesis (such as the Riemann hypothesis) or conjecture (Goldbach's conjecture, that every even number greater than 2 can be written as the sum of two odd primes). Some problems are motivated by practical or theoretical concerns oriented in the real world (applied problems), and others by abstract concerns (for example, what is the distribution of twin primes?). The ones mentioned above are the "big" problems that have been unsolved for decades and whose solution earns the solvers significant notice. But they differ only in scale from the problems encountered in the day-to-day activity of mathematicians. Whether pure or applied, the challenges that ultimately advance our understanding take weeks, months, and often years to solve. This being the case, Halmos argues, students' mathematical experiences should prepare them for tackling such challenges. That is, students should engage in "real" problem solving, learning during their academic careers to work problems of significant difficulty and complexity.

> I do believe that problems are the heart of mathematics, and I hope that as teachers, in the classroom, in seminars, and in the books and articles we write, we will emphasize them more and more, and that we will train our students to be better problem posers and problem solvers than we are. (1980, p. 524)

The mathematician best known for his conceptualization of mathematics as problem solving and for his work in making problem solving the focus of mathematics instruction is Pólya. Indeed, the edifice of problem-solving work erected in the past two decades stands largely on the foundations of his work. The mathematics education community is most familiar with Pólya's work through his (1945/1957) introductory volume *How to Solve It*, in which he introduced the term "modern heuristic" to describe the art of problem solving, and through his subsequent elaborations on the theme in the two-volume sets, *Mathematics and Plausible Reasoning* (1954) and *Mathematical Discovery* (1962, 1965/1981). In fact, Pólya's work on problem solving and "method" was apparent as early as the publication of his and Szegö's (1925) *Problems and Theorems in Analysis*. In this section we focus on the broad mathematical and philosophical themes woven through Pólya's work on problem solving. Details regarding the implementation of heuristic strategies are pursued in the research review.

It is essential to understand Pólya's conception of mathematics as an activity. As early as the 1920s, Pólya had an interest in mathematical heuristics, and he and Szegö included some heuristics (in the form of aphorisms) as suggestions for guiding students' work through the difficult problem sets in *Aufgaben und Lehrsätze aus der Analysis I* (1925). Yet the role of mathematical engagement—of hands-on mathematics, if you will—was central in Pólya's view.

> General rules which could prescribe in detail the most useful discipline of thought are not known to us. Even if such rules could be formulated, they could not be very useful... [for] one must have them assimilated into one's flesh and blood and ready for instant use.... The independent solving of challenging problems will aid the reader far more than the aphorisms which follow, although as a start these can do him no harm. (p. vii)

Part of that engagement, according to Pólya, was the active engagement of discovery, one which takes place in large measure by guessing. Eschewing the notion of mathematics as a formal and formalistic deductive discipline, Pólya argued that mathematics is akin to the physical sciences in its dependence on guessing, insight, and discovery.

> To a mathematician, who is active in research, mathematics may appear sometimes as a guessing game; you have to guess a mathematical theorem before you prove it, you have to guess the idea of the proof before you carry through all the details.
>
> To a philosopher with a somewhat open mind all intelligent acquisition of knowledge should appear sometimes as a guessing game, I think. In science as in everyday life, when faced with a new situation, we start out with some guess. Our first guess may fall short of the mark, but we try it and, according to the degree of success, we modify it more or less. Eventually, after several trials and several modifications, pushed by observations and led by analogy, we may arrive at a more satisfactory guess. The layman does not find it surprising that the naturalist works this way.... And the layman is not surprised to hear that the naturalist is guessing like himself. It may appear a little more surprising to the layman that the mathematician is also guessing. The result of the mathematician's creative work is demonstrative reasoning, a proof, but the proof is discovered by plausible reasoning, by guessing....
>
> Mathematical facts are first guessed and then proved, and almost every passage in this book endeavors to show that such is the normal procedure. If the learning of mathematics has anything to do with the discovery of mathematics, the student must be given some opportunity to do problems in which he first guesses and then proves some mathematical fact on an appropriate level. (1954, pp. 158–160)

For Pólya, mathematical epistemology and mathematical pedagogy are deeply intertwined. Pólya takes it as given that for students to gain a sense of the mathematical enterprise, their experience with mathematics must be consistent with the way mathematics is done. The linkage of epistemology and pedagogy is, as well, the major theme of this chapter. The next section of this chapter elaborates a particular view of mathematical thinking—discussing mathematics as an act of sense-making that is socially constructed and socially transmitted. It argues that students develop their sense of mathematics—and thus how they use mathematics—from their experiences with mathematics (largely in the classroom). It follows that classroom mathematics must mirror this sense of mathematics as a sense-making activity, if students are to come to understand and use mathematics in meaningful ways.

Enculturation and Cognition

An emerging body of literature (Bauersfeld, 1979; Brown, Collins, & Duguid, 1989; Collins, Brown, & Newman, 1989; Greeno, 1989; Lampert, 1990; Lave, 1988; Lave, Smith, & Butler, 1988; Resnick, 1988; Rogoff & Lave, 1984; Schoenfeld, 1989a, 1990b; see especially Nunes, Chapter 22, this volume) conceives of mathematics learning as an inherently social (as well as cognitive) activity, and an essentially constructive activity instead of an absorptive one.

By the mid-1980s, the constructivist perspective, with roots in Piaget's work (1954) and with contemporary research manifestations such as the misconceptions literature (Brown & Burton, 1978; diSessa, 1983; Novak, 1987), was widely accepted in the research community as being well grounded. Romberg and Carpenter (1986) stated the fact bluntly: "The research shows that learning proceeds through *construction*, not *absorption*" (p. 868). The constructivist perspective pervades this handbook as well. However, the work cited in the previous paragraph extends the notion of constructivism from the purely cognitive sphere, where much of the research has been done, to the social sphere. As such, it blends with some theoretical notions from the social literature. Resnick, tracing contemporary work to antecedents in the work of George Herbert Mead (1934) and Lev Vygotsky (1978), states that "several lines of cognitive theory and research point toward the hypothesis that we develop habits and skills of interpretation and meaning construction though a process more usefully conceived of as *socialization* than *instruction*" (1988, p. 39).

The notion of socialization as identified by Resnick (also called enculturation—entering and picking up the values of a community or culture) is central, in that it highlights the importance of perspective and point of view as core aspects of knowledge. The case can be made that a fundamental component of thinking mathematically is having a mathematical point of view, that is, seeing the world in the ways mathematicians do.

> [T]he reconceptualization of thinking and learning that is emerging from the body of recent work on the nature of cognition suggests that becoming a good mathematical problem solver—becoming a good thinker in any domain—may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge. If this is so, we may do well to conceive of mathematics education less as an instructional process (in the traditional sense of teaching specific, well-defined skills or items of knowledge), than as a socialization process. In this conception, people develop points of view and behavior patterns associated with gender roles, ethnic and familial cultures, and other socially defined traits. When we describe the processes by which children are socialized into these patterns of thought, affect, and action, we describe long-term patterns of interaction and engagement in a social environment. (1988, p. 58)

This cultural perspective is well grounded anthropologically, but it is relatively new to the mathematics education literature. The main idea, that point of view is a fundamental determinant of cognition, and that the community to which one belongs shapes the development of one's point of view, is made eloquently by Clifford Geertz.

> Consider... Evans-Pritchard's famous discussion of Azande witchcraft. He is, as he explicitly says but no one seems much to have noticed, concerned with common-sense thought—Zande common-sense thought—as the general background against which the notion of witchcraft is developed....
>
> Take a Zande boy, he says, who has stubbed his foot on a tree stump and developed an infection. Tho boy says it's witchcraft. Nonsense, says Evans-Pritchard, out of his own common-sense tradition: you were merely bloody careless; you should have looked where you were going. I did look where I was going; you have to with so many stumps about, says the boy—*and if I hadn't been witched I would have seen it*. Furthermore, all cuts do not take days to heal, but on the contrary, close quickly, for that is the nature of cuts. But this one festered, thus witchcraft must be involved.
>
> Or take a Zande potter, a very skilled one, who, when now and again one of his pots cracks in the making, cries "witchcraft!" Nonsense! says Evans-Pritchard, who, like all good ethnographers, seems never to learn: of course sometimes pots crack in the making; it's the way of the world. But, says the potter, I chose the clay carefully, I took pains to remove all the pebbles and dirt, I built up the clay slowly and with care, and I abstained from sexual intercourse the night before. And *still* it broke. What else can it be but witchcraft? (1983, p. 78)

Geertz's point is that Evans-Pritchard and the African tribesmen agree on the data (the incidents they are trying to explain), but that their interpretations of what the incidents mean are radically different. Each person's interpretation is derived from his or her own culture and seems common-sensical. The anthropologist in the West and the Africans on their home turf have each developed points of view consonant with the mainstream perspectives of their societies. And those culturally determined (socially mediated) views determine what sense they make of what they see.

The same, it is argued, is true of members of *communities of practice*, groups of people engaged in common endeavors within their own culture. Three such groups include the community of tailors in "Tailors' Alley" in Monrovia, Liberia, studied by Jean Lave (in preparation), the community of practicing mathematicians, and the community that spends its daytime hours in schools. In each case, the "habits and dispositions" (see the quotation from Resnick, above) of community members are culturally defined and have great weight in shaping individual behavior. We discuss the first two here: the third is discussed in the next section. First, Lave's study (which largely inspired the work on cognitive apprenticeship discussed below) examined the apprenticeship system by which Monrovian tailors learn their skills. Schoenfeld summarized Lave's perspective on what "learning to be a tailor" means, as follows:

> *Being a tailor* is more than having a set of tailoring skills. It includes a way of thinking, a way of seeing, and having a set of values and perspectives. In Tailors' Alley, learning the curriculum of tailoring and learning to *be a tailor* are inseparable: the learning takes place in the context of doing real tailors' work, in the community of tailors. Apprentices are surrounded by journeymen and master tailors, from whom they learn their skills—and among whom they live, picking up their values and perspectives as well. These values and perspectives are not part of the formal curriculum of tailoring, but they are a central defining feature

of the environment, and of what the apprentices learn. The apprentice tailors are apprenticing themselves into a *community*, and when they have succeeded in doing so, they have adopted a point of view as well as a set of skills—both of which define them as tailors. [If this notion seems a bit farfetched, think of groups of people such as lawyers, doctors, automobile salesmen, or university professors in our own society. That there are political (and other) stereotypes of these groups indicates that there is more to membership in any of these communities than simply possessing the relevant credentials or skills.] (1989c, pp. 85–86)

Second, there is what might be called "seeing the world through the lens of the mathematician." As illustrations, here are two comments made by the applied mathematician Henry Pollak.

How many saguarro cacti more than 6 feet high are in the state of Arizona? I read that the saguarro is an endangered species. Developers tear them down when they put up new condominiums. So when I visited Arizona two or three years ago I decided to try an estimate. I came up with 10^8. Let me tell you how I arrived at that answer. In the areas where they appear, saguarros seem to be fairly regularly spaced, approximately 50 feet apart. That approximation gave me 10^2 to a linear mile, which implied 10^4 in each square mile. The region where the saguarros grow is at least 50 by 200 miles. I therefore multiplied $10^4 \times 10^4$ to arrive at my final answer. I asked a group of teachers in Arizona for their estimate, and they were at a loss as to how to begin.

If you go into a supermarket, you will typically see a number of checkout counters, one of which is labeled "Express Lane" for x packages or fewer. If you make observations on x, you'll find it varies a good deal. In my home town, the A&P allow six items; the Shop-Rite, eight; and Kings, 10. I've seen numbers vary from 5 to 15 across the country. If the numbers vary that much, then we obviously don't understand what the correct number should be. How many packages should be allowed in an express line? (1987, pp. 260–261)

Both of these excerpts exemplify the habits and dispositions of the mathematician. Hearing that the saguarro is endangered, Pollak almost reflexively asks how many saguarro there might be; he then works out a crude estimate on the basis of available data. This predilection to quantify and model is certainly a part of the mathematical disposition and is not typical of those outside mathematically oriented communities. (Indeed, Pollak notes that neither the question nor the mathematical tools to deal with it seemed natural to the teachers with whom he discussed it.) That disposition is even clearer in the second example, thanks to Pollak's language. Note that Pollak perceives the supermarket as a mathematical context—again, hardly a typical perspective. For most people, the number of items allowed in the express line is simply a matter of the supermarket's prerogative. For Pollak, the number is a variable, and the task of determining the "right" value of that variable is an optimization problem. The habit of seeing phenomena in mathematical terms is also part of the mathematical disposition.

In short, Pollak sees the world from a mathematical point of view. Situations that others might not attend to at all serve for him as the contexts for interesting mathematical problems. The issues he raises in what to most people would be non-mathematical contexts—supermarket check-out lines and desert fields—are inherently mathematical in character. His language ("for x packages or fewer") is that of the mathematician, and his approaches to conceptualizing the problems (optimization for the supermarket problem, estimation regarding the number of cacti) employ typical patterns of mathematical reasoning. There are, of course, multiple mathematical points of view. For a charming and lucid elaboration of many of these, see Davis and Hersh (1981).

Epistemology, Ontology, and Pedagogy Intertwined

In short, the point of the literature discussed in the previous section is that learning is culturally shaped and defined; people develop their understandings of any enterprise from their participation in the "community of practice" within which that enterprise is practiced. The lessons students learn about mathematics in our current classrooms are broadly cultural, extending far beyond the scope of the mathematical facts and procedures (the explicit curriculum) that they study. As Hoffman (1989) points out, this understanding gives added importance to a discussion of epistemological issues. Whether or not one is explicit about one's epistemological stance, he observes, what one thinks mathematics is will shape the kinds of mathematical environments one creates, and thus the kinds of mathematical understandings that one's students will develop.

Here we pursue the epistemological-to-pedagogical link in two ways. First, we perform a detailed exegesis of the selection of "mental arithmetic" exercises from Milne (1897), elaborating the assumptions that underlie it, and the consequences of curricula based on such assumptions. That exegesis is not derived from the literature, although it is consistent with it. The author's intention in performing the analysis is to help establish the context for the literature review, particularly the sections on beliefs and context. Second, we examine some issues in mathematical epistemology and ontology. As Hoffman observes, it is important to understand what doing mathematics is, if one hopes to establish classroom practices that will help students develop the right mathematical point of view. The epistemological explorations in this section establish the basis for the pedagogical suggestions that follow later in the chapter.

On Problems as Practice: An Exegesis of Milne's Problem Set. The selection of exercises from Milne's *Mental Arithmetic* introduced earlier in this chapter has the virtue that it is both antiquated and modern: One can examine it at a distance because of its age, but one will also find its counterparts in almost every classroom around the country. We shall examine it at length.

Recall the first problem posed by Milne: "How much will it cost to plow 32 acres of land at \$3.75 per acre?" His solution was to convert \$3.75 into a fraction of \$10, as follows. "\$3.75 is $\frac{3}{8}$ of \$10. At \$10 per acre the plowing would cost \$320, but since \$3.75 is $\frac{3}{8}$ of \$10, it will cost $\frac{3}{8}$ of \$320, which is \$120." This solution method was then intended to be applied to all of the problems that followed.

It is perfectly reasonable, and useful, to devote instructional time to the technique Milne illustrates. The technique is plausible from a practical point of view, in that there might well be circumstances where a student could most easily do computations of the type demonstrated. It is also quite reasonable from a mathematical point of view. Being able to perceive $A \times B$ as $(r \times C) \times B = r \times (BC)$ when the latter is easier to compute, and carrying out the computation, is a sign that one has developed some understanding of fractions and of multiplicative structures; one would hope that students would develop such understandings in their mathematics instruction. The critique that follows is not based in an objection to the potential value or utility of the mathematics Milne presents, but in the ways in which the topic is treated.

ISSUE 1: FACE VALIDITY. At first glance, the technique illustrated in Milne's problem 52 seems useful, and the solutions to the subsequent problems appear appropriate. As noted above, one hopes that students will have enough number sense to be able to compute 32 × \$3.75 in the absence of paper and pencil. However, there is the serious question as to whether one would really expect students to work the problems the way Milne suggests. In a quick survey as this chapter was being written, the author asked four colleagues to solve problem 52 mentally. Three of the four solutions did convert the ".75" in \$3.75 to a fractional equivalent, but none of the four employed fractions in the way suggested by Milne. The fourth avoided fractions altogether, but also avoided the standard algorithm. Here is what the four did:

- Two of the people converted 3.75 into $3\frac{3}{4}$, and then applied the distributive law to obtain

$$(3\tfrac{3}{4})(32) = (3 + \tfrac{3}{4})(32) = 96 + (\tfrac{3}{4})(32) = 96 + 24 = 120.$$

- One expressed 3.75 as $(4 - \frac{1}{4})$, and then distributed as follows:

$$(4 - \tfrac{1}{4})(32) = 128 - (\tfrac{1}{4})(32) = 128 - 8 = 120.$$

- One noted that 32 is a power of 2. He divided and multiplied by 2's until the arithmetic became trivial:

$$(32)(3.75) = (16)(7.5) = (8)(15) = (4)(30) = 120.$$

In terms of mental economy, we note that each of the methods used is as easy to employ as the one presented by Milne.

ISSUE 2: THE EXAMPLES ARE CONTRIVED TO ILLUSTRATE THE MATHEMATICAL TECHNIQUE AT HAND. In real life one rarely if ever encounters unit prices such as \$6.66$\frac{2}{3}$. (But we commonly see prices such as "3 for \$20.00.") The numbers used in problem 55 and others were clearly selected so that students could successfully perform the algorithm taught in this lesson. On the one hand, choosing numbers in this way makes it easy for students practice the technique. On the other hand, the choice makes the problem itself implausible. Moreover, the problem settings (cords of wood, price of sheep, and so on) are soon seen to be window dressing designed to make the problems appear relevant, but which in fact have no real role in the problem. As such, the artificiality of the examples moves the corpus of exercises from the realm of the practical and plausible to the realm of the artificial.

ISSUE 3: THE EPISTEMOLOGICAL STANCE UNDERLYING THE USE OF SUCH EXERCISE SETS. In introducing Milne's examples, we discussed the pedagogical assumptions underlying the use of such structured problem sets in the curriculum. Here we pursue the ramifications of those assumptions.

Almost all of Western education, particularly mathematics education and instruction, is based on a traditional philosophical perspective regarding epistemology, which is defined as "the theory or science of the method or grounds of knowledge" (*Oxford English Dictionary*, page 884). The fundamental concerns of epistemology regard the nature of knowing and knowledge. "*Know*, in its most general sense, has been defined by some as 'to hold for true or real with assurance and on (what is held to be) an adequate objective foundation' " (*Oxford English Dictionary*, page 1549). In more colloquial terms, the generally held view—often unstated or implicit, but nonetheless powerful—is that what we *know* is what we can justifiably demonstrate to be true; our *knowledge* is the sum total of what we know. That is, one's mathematical knowledge is the set of mathematical facts and procedures one can reliably and correctly use. Jim Greeno pointed out in his review of this chapter that most instruction gives short shrift to the "justifiably demonstrate" part of mathematical knowledge—that it focuses on using techniques, with minimal attention to having students justify the procedures in a deep way. He suggests that if demonstrating is taken in a deep sense, it might be an important curricular objective.

A consequence of the perspective described above is that instruction has traditionally focused on the content aspect of knowledge. Traditionally, one defines what students ought to know in terms of chunks of subject matter and characterizes what a student knows in terms of the amount of content that has been mastered. (The longevity of Bloom's (1956) taxonomies and the presence of standardized curricula and examinations provide clear evidence of the pervasiveness of this perspective.) As natural and innocuous as this view of "knowledge as substance" may seem, it has serious entailments (see Issue 4). From this perspective, "learning mathematics" is defined as mastering, in some coherent order, the set of facts and procedures that comprise the body of mathematics. The route to learning consists of delineating the desired subject-matter content as clearly as possible, carving it into bite-sized pieces, and providing explicit instruction and practice on each of those pieces so that students master them. From the content perspective, the whole of a student's mathematical understanding is precisely the sum of these parts.

> Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly (Ball, 1988; Schoenfeld, 1985b; Stodolsky, 1985). These cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth *is determined* when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing. (Lampert, 1990, p. 31)

These assumptions play out clearly in the selection from Milne. The topic to be mastered is a particular, rather narrow technique. The domain of applicability of the technique is made clear: Initially it applies to decimals that can be written as $(a/b) \times 10$, and then the technique is extended to apply to decimals that can be written as $(a/b) \times 100$. Students are constrained to use this technique, and when they master it, they move on to the next. For many students, experience with problem sets of this type is their sole encounter with mathematics.

Issue 4: The cumulative effects of such exercise sets. As Lampert notes, students' primary experience with mathematics—the grounds upon which they build their understanding of the discipline—is their exposure to mathematics in the classroom. The impression given by this set of exercises, and thousands like it that students work in school, is that there is one right way to solve the given set of problems—the method provided by the text or instructor. As indicated in the discussion of Issue 1, this is emphatically not the case; there are numerous ways to arrive at the answer. However, in the given instructional context only one method appears legitimate. There are numerous consequences to repeated experiences of this type.

One consequence of experiencing the curriculum in bite-size pieces is that students learn that answers and methods to problems will be provided to them; the students are not expected to figure out the methods by themselves. Over time, most students come to accept their passive role and to think of mathematics as "handed down" by experts for them to memorize (Carpenter, Lindquist, Matthews, & Silver, 1983; National Assessment of Educational Progress, 1983).

A second consequence of the nonproblematic nature of these "problems" is that students come to believe that in mathematics, (1) one should have a ready method for the solution of a given problem, and (2) the method should produce an answer to the problem in short order (Carpenter et al., 1983; National Assessment of Educational Progress, 1983; Schoenfeld, 1988, 1989b). In the 1983 National Assessment, about half of the students surveyed agreed with the statement, "Learning mathematics is mostly memorizing." Three quarters of the students agreed with the statement, "Doing mathematics requires lots of practice in following rules," and nine students out of ten agreed with the statement, "There is always a rule to follow in solving mathematics problems" (NAEP, 1983, pp. 27–28). As a result of holding such beliefs, students may not even attempt problems for which they have no ready method, or may curtail their efforts after only a few minutes without success.

More importantly, the methods imposed on students by teacher and texts may appear arbitrary and may contradict the alternative methods that the students have tried to develop for themselves. For example, all of the problems given by Milne, and more generally, in most mathematics, can be solved in a variety of ways. However, only one method was sanctioned in Milne's text. Recall also that some of the problems were clearly artificial, negating the practical virtues of the mathematics. After consistent experiences of this type, students may simply give up trying to make sense of the mathematics. They may consider the problems to be exercises of little meaning, despite their applied cover stories; they may come to believe that mathematics is not something they can make sense of, but rather something almost completely arbitrary (or at least whose meaningfulness is inaccessible to them) and which must thus be memorized without looking for meaning—if they can cope with it at all (Lampert, 1990; Stipek & Weisz, 1981; Tobias, 1978). More detail is given in the section on belief systems.

The Mathematical Enterprise. Over the past two decades, there has been a significant change in the face of mathematics (its scope and the very means by which it is carried out) and in the community's understanding of what it is to know and do mathematics. A series of recent articles and reports (Hoffman, 1989; National Research Council, 1989; Steen, 1988) attempts to characterize the nature of contemporary mathematics and to point to changes in instructions that follow from the suggested reconceptualization. The main thrust of this reconceptualization is to think of mathematics, broadly, as "the science of patterns."

> *MATHEMATICS . . . searching for patterns*
> Mathematics reveals hidden patterns that help us understand the world around us. Now much more than arithmetic and geometry, mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, and proof; and with mathematical models of natural phenomena, of human behavior, and of social systems.
>
> The cycle from data to deduction to application recurs everywhere mathematics is used, from everyday household tasks such as planning a long automobile trip to major management problems such as scheduling airline traffic or managing investment portfolios. The process of "doing" mathematics is far more than just calculation or deduction; it involves observation of patterns, testing of conjectures, and estimation of results.
>
> As a practical matter, mathematics is a science of pattern and order. Its domain is not molecules or cells, but numbers, chance, form, algorithms, and change. As a science of abstract objects, mathematics relies on logic rather than observation as its standard of truth, yet employs observation, simulation, and even experimentation as a means of discovering truth. (National Research Council, 1989, p. 31)

In this quotation there is a major shift from the traditional focus on the content aspect of mathematics discussed above (where attention is focused primarily on the mathematics one "knows"), to the process aspects of mathematics (what is known as "doing mathematics"). Indeed, content is mentioned only in passing, while modes of thought are specifically highlighted.

> In addition to theorems and theories, mathematics offers distinctive modes of thought which are both versatile and powerful, including modeling, abstraction, optimization, logical analysis, inference from data, and use of symbols. Experience with mathematical modes of thought builds mathematical power—a capacity of mind of increasing value in this technological age that enables one to read critically, to identify fallacies, to detect bias, to assess risk, and to suggest alternatives. Mathematics empowers us to understand better the information-laden world in which we live. (National Research Council, 1989, pp. 31–32)

One main change, then, is that there is a large focus on process rather than on mathematical content in describing both what mathematics is and what one hopes students will learn from studying it. In this sense, mathematics appears much more like science than it would if one focused solely on the subject matter. Indeed, the "science of patterns" may seem so broad a definition as to obscure the mathemat-

irregular
type
size

-gular
-ype
size

ical core contained therein. What makes it mathematical is the domain over which the abstracting or patterning is done and the choice of tools and methods typically employed. To repeat from the introductory definition, mathematics consists of "systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically (pure mathematics) or models of systems abstracted from real world objects (applied mathematics). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation" (Schoenfeld, in preparation).

A second main change, reflected in the statement that "mathematics relies on logic rather than observation as its standard of truth, yet employs observation, simulation, and even experimentation as a means of discovering truth" (National Research Council, 1989, p. 32) reflects a growing understanding of mathematics as an empirical discipline of sorts, one in which mathematical practitioners gather data in the same ways that scientists do. This theme is seen in the writings of Lakatos (1977, 1978), who argued that mathematics does not, as it often appears, proceed inexorably and inevitably by deduction from a small set of axioms; rather that the community of mathematicians decides what is axiomatic, in effect making new definitions if the ones that have been used turn out to have untoward consequences. A third change is that doing mathematics is increasingly coming to be seen as a social and collaborative act. Steen's (1988) examples of major progress in mathematics—in number theory (the factorization of huge numbers and prime testing, requiring collaborative networks of computers), in the Nobel Prize-winning application of the Radon Transform to provide the mathematics underlying the technology for computer-assisted tomography (CAT) scans, and in the solution of some recent mathematical conjectures such as the four-color theorem—are all highly collaborative efforts. Collaboration, on the individual level, is discussed with greater frequency in the "near mathematical" literature, as in these two excerpts from Albers and Alexanderson's (1985) *Mathematical People: Profiles and Interviews*. Peter Hilton lays out the benefits of collaboration as follows:

> First I must say that I do enjoy it. I very much enjoy collaborating with friends. Second, I think it is an efficient thing to do because... if you are just working on your own [you may] run out of steam.... But with two of you, what tends to happen is that when one person begins to feel a flagging interest, the other one provides the stimulus.... The third thing is, if you choose people to collaborate with who somewhat complement rather than duplicate the contribution that you are able to make, probably a better product results. (p. 141)

Persi Diaconis says the following:

> There is a great advantage in working with a great co-author. There is excitement and fun, and it's something I notice happening more and more in mathematics. Mathematical people enjoy talking to each other.... Collaboration forces you to work beyond your normal level. Ron Graham has a nice way to put it. He says that when you've done a joint paper, both co-authors do 75% of the work, and that's about right.... Collaboration for me means enjoying talking and explaining, false starts, and the interaction of personalities. It's a great, great joy to me. (pp. 74–75)

For these individuals, and for those engaged in the kinds of collaborative efforts discussed by Steen, membership in the mathematical community is without question an important part of their mathematical lives. However, there is an emerging epistemological argument suggesting that mathematical collaboration and communication have a much more important role than indicated by the quotes above. According to that argument, ***membership in a community of mathematical practice is part of what constitutes mathematical thinking and knowing.*** Greeno notes that this idea takes some getting used to.

> The idea of a [collaborative] practice contrasts with our standard ways of thinking about knowledge. We generally think of knowledge as some content in someone's mind, including mental structures and procedures. In contrast, a practice is an everyday activity, carried out in a socially meaningful context in which activity depends on communication and collaboration with others and knowing how to use the resources that are available in the situation...
>
> An important [philosophical and historical] example has been contributed by Kitcher (1984). Kitcher's goal was to develop an epistemology of mathematics. The key concept in his epistemology is an idea of a mathematical practice, and mathematical knowledge is to be understood as knowledge of mathematical practice. A mathematical practice includes understanding of the language of mathematical practice, and the results that are currently accepted as established. It also includes knowledge of the currently important questions in the field, the methods of reasoning that are taken as valid ways of establishing new results, and metamathematical views that include knowledge of general goals of mathematical research and appreciation of criteria of significance and elegance. (1988, pp. 24–25)

That is, "having a mathematical point of view" and "being a member of the mathematical community" are central aspects of having mathematical knowledge. Schoenfeld makes the case as follows:

> I remember discussing with some colleagues, early in our careers, what it was like to be a mathematician. Despite obvious individual differences, we had all developed what might be called the mathematician's point of view—a certain way of thinking about mathematics, of its value, of how it is done, etc. What we had picked up was much more than a set of skills; it was a way of viewing the world, and our work. We came to realize that we had undergone a process of acculturation, in which we had become members of, and had accepted the values of, a particular community. As the result of a protracted apprenticeship into mathematics, we had become mathematicians in a deep sense (by dint of world view) as well as by definition (what we were trained in, and did for a living). (1987a, p. 213)

The epistemological perspective discussed here dovetails closely with the enculturation perspective discussed earlier in this chapter. Recall Resnick's (1989) observation that "becoming a good mathematical problem solver—becoming a good thinker in any domain—may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge" (p. 58). The critical observation in both the mathematical and the school contexts is that one develops one's point of view by the process of acculturation, by becoming a member of the particular community of practice.

Goals for Instruction, and a Pedagogical Imperative

The Mathematical Association of America's Committee on the Teaching of Undergraduate Mathematics recently issued a *Source Book for College Mathematic Teaching* (Schoenfeld, 1990a). The *Source Book* begins with a statement of goals for instruction, which seem appropriate for discussion here.

> *Goals for Mathematics Instruction*
> Mathematics instruction should provide students with a sense of the discipline—a sense of its scope, power, uses, and history. It should give them a sense of what mathematics is and how it is done, at a level appropriate for the students to experience and understand. As a result of their instructional experiences, students should learn to value mathematics and to feel confident in their ability to do mathematics.
>
> Mathematics instruction should develop students' understanding of important concepts in the appropriate core content.... Instruction should be aimed at conceptual understanding rather than at mere mechanical skills, and at developing in students the ability to apply the subject matter they have studied with flexibility and resourcefulness.
>
> Mathematics instruction should provide students the opportunity to explore a broad range of problems and problem situations, ranging from exercises to open-ended problems and exploratory situations. It should provide students with a broad range of approaches and techniques (ranging from the straightforward application of the appropriate algorithmic methods to the use of approximation methods, various modeling techniques, and the use of heuristic problem-solving strategies) for dealing with such problems.
>
> Mathematics instruction should help students to develop what might be called a "mathematical point of view"—a predilection to analyze and understand, to perceive structure and structural relationships, to see how things fit together. (Note that those connections may be either pure or applied.) It should help students develop their analytical skills, and the ability to reason in extended chains of argument.
>
> Mathematics instruction should help students to develop precision in both written and oral presentation. It should help students learn to present their analyses in clear and coherent arguments reflecting the mathematical style and sophistication appropriate to their mathematical levels. Students should learn to communicate with us and with each other, using the language of mathematics.
>
> Mathematics instruction should help students develop the ability to read and use text and other mathematical materials. It should prepare students to become, as much as possible, independent learners, interpreters, and users of mathematics. (Schoenfeld, 1990a, p. 2)

In light of the discussion from *Everybody Counts*, we would add the following to the second goal: Mathematics instruction should help students develop mathematical power, including the use of specific mathematical modes of thought that are both versatile and powerful, including modeling, abstraction, optimization, logical analysis, inference from data, and use of symbols.

If these are plausible goals for instruction, one must ask what kinds of instruction might succeed at producing them. The literature reviewed in this part of the chapter, in particular the literature on socialization and epistemology, produces what is in essence a pedagogical imperative:

> *If one hopes for students to achieve the goals specified here—in particular, to develop the appropriate mathematical habits and dispositions of interpretation and sense-making as well as the appropriately mathematical modes of thought—then the communities of practice in which they learn mathematics must reflect and support those ways of thinking. That is, classrooms must be communities in which mathematical sense-making, of the kind we hope to have students develop, is practiced.*

A FRAMEWORK FOR EXPLORING MATHEMATICAL COGNITION

The Framework

The first part of this chapter focused on the mathematical enterprise—what *Everybody Counts* calls "doing" mathematics. Here we focus on the processes involved in thinking mathematically—the psychological support structure for mathematical behavior. The main focus of our discussion is on developments over the past quarter century. It would seem short-sighted to ignore the past 2,000 years of philosophy and psychology related to mathematical thinking and problem solving, so we begin with a brief historical introduction (see Peters, 1962, or Watson, 1978, for detail) to establish the context for the discussion of contemporary work and explain why the focus, essentially de novo, is on the past few decades. For ease of reference, we refer to the enterprise under the umbrella label "psychological studies," including contributions from educational researchers, psychologists, social scientists, philosophers, and cognitive scientists, among others. General trends are discussed here, with details regarding mathematical thinking given in the subsequent sections.

The roots of contemporary studies in thinking and learning can be traced to the philosophical works of Plato and Aristotle. More directly, Descartes's (1952) *Rules for the Direction of the Mind* can be seen as the direct antecedent of Pólya's (1945, 1954, 1981) prescriptive attempts at problem solving. However, the study of the mind and its workings did not turn into an empirical discipline until the late 19th century. The origins of that discipline are usually traced to the opening of Wundt's laboratory in Leipzig, Germany, in 1879. "Wundt was the first modern psychologist—the first person to conceive of experimental psychology as a science.... The methodological prescriptive allegiances of Wilhelm Wundt are similar to those of the physiologists from whom he drew inspiration....[H]e subscribed to methodological objectivism in that he attempted to quantify experience so that others could repeat his procedures... Since the combination of introspection and experiment was the method of choice, Wundt fostered empiricism" (Watson, 1978, p. 292). Wundt (1904) and colleagues employed the methods of experimentation and introspection (self-reports of intellectual processes) to gather data about the workings of mind. These methods may have gotten psychology off to an empirical start, but they soon led to difficulties; members of different laboratories reported different kinds of introspections (corresponding to the theories held in those laboratories), and there were significant problems with both reliability and replicability of the research findings.

American psychology's origins at the the turn of the century were more philosophical, tied to pragmatism and functionalism. William James is generally considered the first major American psychologist, and his (1890) *Principles of Psychology* is an exemplar of the American approach. James's student, E. L. Thorndike, began with animal studies and moved to studies of human cognition. Thorndike's work, in particular, had great impact on theories of mathematical cognition.

One of the major rationales for the teaching of mathematics, dating back to Plato, was the notion of mental discipline. Simply put, the idea is that those who are good at mathematics tend to be good thinkers; those who are trained in mathematics learn to be good thinkers. As exercise and discipline train the body, the theory went, the mental discipline associated with doing mathematics trains the mind, making one a better thinker. Thorndike's work challenged this hypothesis. He offered experimental evidence that transfer of the type suggested by the notion of mental discipline was minimal (Thorndike & Woodworth, 1901) and argued that the benefits attributed to the study of mathematics were correlational: Students with better reasoning skills tended to take mathematics courses (Thorndike, 1924). His research, based in animal and human studies, put forth the "law of effect," which says in essence "you get good at what you practice, and there isn't much transfer." His "law of exercise" gave details of the ways (recency and frequency effects) learning took place as a function of practice. As Peters notes, "Few would object to the first, at any rate, of these two laws, as a statement of a necessary condition of learning; it is when they come to be regarded as sufficient conditions that uneasiness starts" (1963, p. 695).

Unfortunately, that sufficiency criterion grew and held sway for quite some time. On the continent, Wundt's introspectionist techniques were shown to be methodologically unreliable, and the concept of mentalism came under increasing attack. In Russia, Pavlov (1924) achieved stunning results with conditioned reflexes, his experimental work requiring no concept of mind at all. Finally, mind, consciousness, and all related phenomena were banished altogether by the behaviorists. John Watson (1930) was the main exponent of the behaviorist stance; and B. F. Skinner (1974) was a zealous adherent. The behaviorists were vehement in their attacks on mentalism and provoked equally strong counter-reactions.

> John Watson and other behaviorists led a fierce attack, not only on introspectionism, but also on any attempt to develop a theory of mental operations. Psychology, according to the behaviorists, was to be entirely concerned with external behavior and not to try to analyze the workings of the mind that underlay this behavior:
>
> > Behaviorism claims that consciousness is neither a definite nor a usable concept. The behaviorist, who has been trained always as an experimentalist, holds further that belief in the existence of consciousness goes back to the ancient days of superstition and magic. (Watson, 1930, p. 2)
> >
> > ...The behaviorist began his own conception of the problem of psychology by sweeping aside all medieval conceptions. He dropped from his scientific vocabulary all subjective terms such as sensation, perception, image, desire, purpose, and even thinking and emotion as they were subjectively defined. (Watson, 1930, p. 5)
>
> The behaviorist program and the issues it spawned all but eliminated any serious research in cognitive psychology for 40 years. The rat supplanted the human as the principal laboratory subject, and psychology turned to finding out what could be learned by studying animal learning and motivation. (Anderson, 1985, p. 7)

While behaviorism held center stage, alternate perspectives were in the wings. Piaget's work (1928, 1930, 1971), while rejected by his American contemporaries as being unrigorous, established the basis for the constructivist perspective, the now well-established position that individuals do not perceive the world directly, but that they perceive interpretations of it mediated by the interpretive frameworks they have developed. The Gestaltists, particularly Duncker, Hadamard, and Wertheimer, were interested in higher-order thinking and problem solving. The year 1945 was a banner year for the Gestaltists. Duncker's (1945) monograph *On Problem Solving* appeared in English, as did Hadamard's (1945) *Essay on the Psychology of Invention in the Mathematical Field* [which provides a detailed exegesis of Poincare's (1913) description of his discovery of the structure of Fuchsian functions], and Wertheimer's (1945/1959) *Productive Thinking*, which includes Wertheimer's famous discussion of the "parallelogram problem" and an interview with Einstein on the origins of relativity theory. These works all continued the spirit of Graham Wallas's (1926) *The Art of Thought*, in which Wallas codified the four-step Gestalt model of problem solving: saturation, incubation, inspiration, and verification. The Gestaltists, especially Wertheimer, were concerned with structure and deep understanding. Unfortunately their primary methodological tool was introspection, and they were vulnerable to attack on the basis of the methodology's lack of reliability and validity. (They were also vulnerable because they had no plausible theory of mental mechanism, while the behaviorists could claim that stimulus-response chains were modeled on neuronal connections.) To cap off the year 1945, Pólya's *How to Solve It*—compatible with the Gestaltists' work, but more prescriptive, à la Descartes, in flavor—appeared as well.

The downfall of behaviorism and the renewed advent of mentalism, in the form of the information processing approach to cognition, began in the mid-1950s (see Newell & Simon, 1972, pp. 873 ff. for detail). The development of artificial intelligence programs to solve problems, such as Newell & Simon's (1972) "General Problem Solver," hoist the behaviorists by their own petard.

> The simulation models of the 1950s were offspring of the marriage between ideas that had emerged from symbolic logic and cybernetics, on the one hand, and Würzburg and Gestalt psychology, on the other. From logic and cybernetics was inherited the idea that information transformation and transmission can be described in terms of the behavior of formally described symbol manipulation systems. From Würzburg and Gestalt psychology were inherited the ideas that long-term memory is an organization of directed associations and that problem solving is a process of directed goal-oriented search. (Simon, 1979, pp. 364–365)

Note that the information-processing work discussed by Simon met the behaviorists' standards in an absolutely incontrovertible way: Problem-solving programs (simulation models and artificial intelligence programs) produced problem-solving behavior, and all the workings of the program were out in the open for inspection. At the same time, the theories and methodologies of the information-processing school were fundamentally mentalistic—grounded in the theories of mentalistic psychology, and using observations of humans engaged in problem solving to infer the structure of their (mental) problem-solving strategies. Although it was at least a decade before such work had an impact on mainstream experimental psychology (Simon, 1979), and it was as late as 1980 that Simon and colleagues (Ericsson & Simon, 1980) were writing review articles hoping to legitimize the use of out-loud problem-solving protocols, an emphasis on cognitive processes emerged, stabilized, and began to predominate in psychological studies of mind.

Early work in the information-processing (IP) tradition was extremely narrow in focus, partly because of the wish to have clean, scientific results. For many, the only acceptable test of a theory was a running computer program that did what its author said it should. Early IP work often focused on puzzle domains (such as the Tower of Hanoi problem and its analogues), with the rationale that in such simple domains one could focus on the development of strategies, and then later move to semantically rich domains. As the tools were developed, studies moved from puzzles and games (logic, cryptarithmetic, and chess, for example) to more open-ended tasks, focusing on textbook tasks in domains such as physics and mathematics (and later, in developing expert systems in medical diagnosis, mass spectroscopy, and other areas). Nonetheless, work in the IP tradition remained quite narrow for some time. The focus was on the "architecture of cognition" (and machines): the structure of memory, of knowledge representations, of knowledge retrieval mechanisms, and of problem-solving rules.

During the same time period (the first paper on metamemory by Flavell, Friedrichs, & Hoyt appeared in 1970; the topic peaked in the mid-to-late 1980s) metacognition became a major research topic. Here too, the literature is quite confused. In an early paper, Flavell characterized the term as follows:

> Metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them, e.g. the learning-relevant properties of information or data. For example, I am engaging in metacognition... if I notice that I am having more trouble learning A than B; if it strikes me that I should double-check C before accepting it as a fact; if it occurs to me that I should scrutinize each and every alternative in a multiple-choice task before deciding which is the best one.... Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of those processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete [problem solving] goal or objective. (1976, p. 232)

This kitchen-sink definition includes a number of categories which have since been separated into more functional categories for exploration: (1) individuals' declarative knowledge about their cognitive processes, (2) self-regulatory procedures, including monitoring and "on-line" decision-making, and (3) beliefs and affects and their effects on performance. (Through the early 1980s, the cognitive and affective literatures were separate and unequal. The mid-1980s saw a rapprochement, with the notion of beliefs extending the scope of the cognitive inquiries to be at least compatible with those of the affective domain. Since then, the enculturation perspective discussed earlier has moved the two a bit closer.) These subcategories are considered in the framework elaborated below.

Finally, the tail end of the 1980s saw a potential unification of aspects of what might be called the cognitive and social perspectives on human behavior, in the theme of enculturation. The minimalist version of this perspective is that learning is a social act, taking place in a social context; that one must consider learning environments as cultural contexts and learning as a cultural act. (The maximal version, yet to be realized theoretically, is a unification that allows one to see what goes on "inside the individual head" and "distributed cognition" as aspects of the same thing.) Motivated by Lave's work on apprenticeship (1988, in preparation), Collins et al. (1989) abstracted common elements from productive learning environments in reading (Palincsar & Brown, 1984), writing (Scardamalia & Bereiter, 1983) and mathematics (Schoenfeld, 1985a). Across the case studies they found a common, broad conceptualization of domain knowledge that included not only the specifics of domain knowledge, but also an understanding of strategies and aspects of metacognitive behavior. In addition, they found that all three programs had aspects of "the culture of expert practice," in that the environments were designed to take advantage of social interactions to have students experience the gestalt of the discipline in ways comparable to the ways that practitioners do.

In general, research in mathematics education followed a similar progression of ideas and methodologies. Through the 1960s and 1970s, correlational, factor-analytic, and statistical "treatment A vs. treatment B" comparison studies predominated in the "scientific" study of thinking, learning, and problem solving. By the mid-1970s, however, researchers expressed frustration at the limitations of the kinds of contributions that could, in principle, be made by such studies of mathematical behavior. For example, Kilpatrick (1978) compared the research methods prevalent in the United States at the time with the kinds of qualitative research being done in the Soviet Union by Krutetskii (1976) and his colleagues. The American research, he claimed, was "rigorous" but somewhat sterile. In the search for experimental rigor, researchers had lost touch with truly meaningful mathematical behavior. In contrast, the Soviet studies of mathematical abilities were decidedly unrigorous, if not unscientific, but they focused on behavior and abilities that had face validity as important aspects of mathematical thinking. Kilpatrick suggested that the research community might do well to broaden the scope of its inquiries and methods.

Indeed, researchers in mathematics education turned increasingly to process-oriented studies in the late 1970s and 1980s. Much of the process-oriented research was influenced by the trends in psychological work described above, but it also had its own special character. As noted above, psychological research tended to focus on "cognitive architecture"—studies of the structure of memory, of representations, and so forth. From a psychological point of view, mathematical tasks were attractive as settings for such research because of their (ostensibly) formal, context-independent nature. That is, topics from literature or history might be "contaminated" by real-world knowledge, a fact that would make it difficult to control precisely what students brought to, or learned in, experimental settings. But purely formal topics from mathematics (e.g., the algorithm for base 10 addition and subtraction or the rules for solving linear equations in one variable) could be taught as purely formal manipulations, and thus one could avoid the difficulties of "contamination." In an early information-processing study of problem solving, for example, Newell and Simon (1972) analyzed the behavior of students solving problems in symbolic logic. From their observations, they abstracted successful patterns of symbol manipulation and wrote them as computer programs. However, Newell and Simon's sample explicitly excluded any subjects who knew the meanings of the symbols (e.g., that "P → Q" means "if P is true, then Q is true"), because their goal was to find productive modes of symbol manipulation without understanding the symbols, since the computer programs they intended to write wouldn't be able to reason on the basis of those meanings. In contrast, of course, the bottom line for most mathematics educators is to have students develop an understanding of the procedures and their meanings. Hence, the IP work took on a somewhat different character when adapted for the purposes of mathematics educators.

The state of the art in the early and late 1980s, respectively, can be seen in two excellent summary volumes—Silver's (1985) *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* and Charles and Silver's (1988) *The Teaching and Assessing of Mathematical Problem Solving*. Silver's volume was derived from a conference held in 1983, which brought together researchers from numerous disciplines to discuss results and directions for research in problem solving. Some confusion, a great deal of diversity, and a flowering of potentially valuable perspectives are evident in the volume. There was confusion, for example, about baseline definitions of "problem solving." Kilpatrick (1985), for example, gave a range of definitions and examples that covered the spectrum discussed in the first part of this chapter. And either explicitly or implicitly, that range of definitions was exemplified in the chapters of the book. At one end of the spectrum, Carpenter (1985) began his chapter with a discussion of the following problem: "James had 13 marbles. He lost 8 of them. How many marbles does he have left?" Carpenter notes that "such problems frequently are not included in discussions of problem solving because they can be solved by the routine application of a single arithmetic operation. A central premise of this paper is that the solutions of these problems, particularly the solutions of young children, do in fact involve real problem-solving behavior" (p. 17). Heller and Hungate (1985) implicitly take their definition of problem solving to mean being able to solve the exercises at the end of a standard textbook chapter, as does Mayer. At the other end of the spectrum, "the fundamental importance of epistemological issues (e.g., beliefs, conceptions, and misconceptions) is reflected in the papers by Jim Kaput, Richard Lesh, Alan Schoenfeld, and Mike Shaughnessy (p. ix.)". Those chapters took a rather broad view of problem solving and mathematical thinking. Similarly, the chapters reveal a great diversity of methods and their productive application to issues related to problem solving. Carpenter's chapter presents detailed cross-sectional data on children's use of various strategies for solving word problems of the type discussed above. Heller and Hungate worked within the "expert-novice" paradigm for identifying the productive behavior of competent problem solvers and using such behavior as a guide for instruction for novices. Mayer discussed the application of schema theory, again within the expert-novice paradigm. Kaput discussed fundamental issues of representation and their role in understanding, Shaughnessy discussed misconceptions, and Schoenfeld explained the roles of metacognition and beliefs. Alba Thompson (1985) studied teacher beliefs and their effects on instruction. There was also diversity in methodology: experimental methods, expert-novice studies, clinical interviews, protocol analyses, and classroom observations, among others. The field had clearly flowered, and there was a wide range of new work.

The Charles and Silver volume (1988) reflects a maturing of and continued progress in the field. By the end of the decade, most of the methodologies and perspectives tentatively explored in the Silver volume had been studied at some length, with the result that they had been contextualized in terms of just what they could offer in terms of explaining mathematical thinking. For example, the role of information-processing approaches and the expert-novice paradigm could be seen as providing certain kinds of information about the organization and growth of individual knowledge, but as illuminating only one aspect of a much larger and more complex set of issues. With more of the methodological tools in place, it became possible to take a broad view once again, focusing, for example, on history (the Stanic and Kilpatrick chapter in the Charles and Silver volume) and epistemology as grounding contexts for explorations into mathematical thinking. In the Charles and Silver volume, one sees the theme of social interactions and enculturation emerging as central concerns, while in the earlier Silver volume such themes were noted but put aside as "things we aren't really ready to deal with." What one sees is the evolution of overarching frameworks, such as cognitive apprenticeship, that deal with individual learning in a social context. That social theme is explored in the work of Greeno (1988), Lave et al. (1988), and Resnick (1988), among others. There is not at present anything resembling a coherent explanatory frame—that is, a principled explanation of how the varied aspects of mathematical thinking and problem solving fit together. However, there does appear to be an emerging consensus about the necessary scope of inquiries into mathematical thinking and problem solving. Although the fine detail varies [Collins et al. (1989) subsume the last two categories under a general discussion of "culture"; Lester, Garofalo, & Kroll (1989) subsume problem-solving strategies under the knowledge base, while maintaining separate categories for belief and affect], there appears to be general agreement on the importance of these five aspects of cognition:

- The knowledge base
- Problem-solving strategies
- Monitoring and control
- Beliefs and affects
- Practices

These five categories provide the framework employed in the balance of the review.

The Knowledge Base

Research on human cognitive processes over the past quarter century has focused on the organization of, and access to, information contained in memory. In the crudest terms, the underlying issues have been how information is organized and stored in the head; what comprises understanding; and how individuals have access to relevant information. The mainstream idea is that humans are information processors and that in their minds humans construct symbolic representations of the world. According to this view, thinking about and acting in the world consist, respectively, of operating mentally on those representations and taking actions externally that correspond to the results of our minds' internal workings. While these are the mainstream positions—and the ones elaborated below—it should be noted that all of them are controversial. There is, for example, a theoretical stance regarding distributed cognition (Pea, 1989), which argues that it is inappropriate to locate knowledge "in the head"—that knowledge resides in communities and their artifacts and in interactions between individuals and their environments (which include other people). The related concept of situated cognition (see, for example, Barwise & Perry, 1983; Brown et al., 1989; Lave & Wenger, 1989) is based on the underlying assumption that mental representations are not complete and that thinking exploits the features of the world in which one is embedded, rather than operating on abstractions of it. Moreover, even if one accepts the notion of internal cognitive representation, there are multiple perspectives regarding the nature and function of representations (see, for example, Janvier, 1987, for a collection of papers regarding perspectives on representations in mathematical thinking; for a detailed elaboration of such issues within the domain of algebra, see Wagner & Kieran, 1989, especially the chapter by Kaput), or of what "understanding" might be. (For a detailed elaboration of such themes with regard to elementary mathematics, see Putnam, Lampert, & Peterson, 1989.) Hence, the sequel presents what might be considered "largely agreed upon" perspectives.

Suppose a person finds him or herself in a situation that calls for the use of mathematics, either for purposes of interpretation (mathematizing) or problem solving. In order to understand the individual's behavior—which options are pursued, in which ways—one needs to know what mathematical tools the individual has at his or her disposal. Simply put, the issues related to the individual's knowledge base are: What information relevant to the mathematical situation or problem at hand does he or she possess? And how is that information accessed and used?

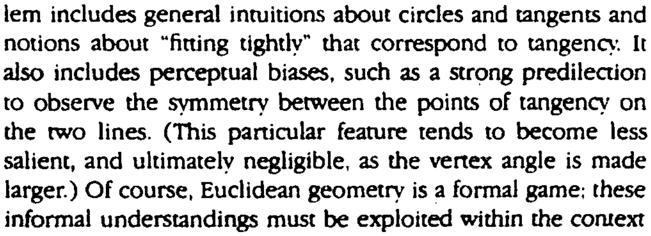
Although these two questions appear closely related, they are, in a sense, almost independent. By way of analogy, consider the parallel questions with regard to the contents of a library: What's in it, and how do you gain access to the contents? The answer to the first question is contained in the catalogue: a list of books, records, tapes, and other materials the library possesses. The contents are what interest you if you have a particular problem or need particular resources. How the books are catalogued or how you gain access to them is somewhat irrelevant (especially if the ones you want aren't in the catalogue). On the other hand, once you are interested in finding and using something listed in the catalogue, the situation changes. How the library actually works becomes critically important: Procedures for locating a book on the shelves, taking it to the desk, and checking it out must be understood. Note, incidentally, that these procedures are largely independent of the contents of the library. One would follow the same set of procedures for accessing any two books in the general collection.

The same holds for assessing the knowledge base an individual brings to a problem-solving situation. In analyses of problem-solving performance, for example, the central issues most frequently deal with what individuals know (the contents of memory) and how that knowledge is deployed. In assessing decision-making during problem solving, for instance, one needs to know what options problem solvers had available. Did they fail to pursue particular options because they overlooked them or because they didn't know of their existence? In the former case, the difficulty might be metacognitive or not seeing the right "connections"; in the latter case, it is a matter of not having the right tools. From the point of view of the observer or experimenter trying to understand problem-solving behavior, then, a major task is the delineation of the knowledge base of individuals who confront the given problem-solving tasks. It is important to note that in this context, the knowledge base may contain things that are not true. Individuals bring misconceptions and misunderstood facts to problem situations; it is essential to understand that *those* are the tools they work with.

The Knowledge Inventory (Memory Contents). Broadly speaking, aspects of the knowledge base relevant for competent performance in a domain include informal and intuitive knowledge about the domain; facts, definitions, and the like; algorithmic procedures; routine procedures; relevant competencies; and knowledge about the rules of discourse in the domain. (This discussion is abstracted from pages 54–61 of Schoenfeld, 1985a.) Consider, for example, the resources an individual might bring to the following problem:

Problem

You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using a straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines.



Informal knowledge an individual might bring to the problem includes general intuitions about circles and tangents and notions about "fitting tightly" that correspond to tangency. It also includes perceptual biases, such as a strong predilection to observe the symmetry between the points of tangency on the two lines. (This particular feature tends to become less salient, and ultimately negligible, as the vertex angle is made larger.) Of course, Euclidean geometry is a formal game; these informal understandings must be exploited within the context of the rules for constructions. As noted above, the facts, definitions, and algorithmic procedures the individual brings to the

TABLE 15–1. Part of the Inventory of an Individual's Resources for Working the Construction Problem

	Degree of Knowledge	*of*	*facts*	*and*	*procedures*
	Does the student:				
5.1	a. know nothing about b. know about the existence of, but nothing about the details of c. partially recall or suspect the details, but with little certainty d. confidently believe		the tangent to a circle is perpendicular to the radius drawn to the point of tangency (true)		a (correct) procedure for bisecting an angle
15.1			any two constructible loci suffice to determine the location of a point (true with qualifications)		a (correct) procedure for dropping a perpendicular to a line from a point
15.2			the center of an inscribed circle in a triangle lies at the intersection of the medians (false)		an (incorrect) procedure for erecting a perpendicular to a line through a given point on that line

problem situation may or may not be correct; they may be held with any degree of confidence from absolute (but possibly incorrect) certainty to great uncertainty. Part of this aspect of the knowledge inventory is outlined in Table 15.1.

Routine procedures and relevant competencies differ from facts, definitions, and algorithmic procedures in that they are somewhat less cut-and-dried. Facts are right or wrong. Algorithms, when applied correctly, are guaranteed to work. Routine procedures are likely to work, but with no guarantees. For example, the problem above, although stated as a construction problem, is intimately tied to a proof problem. One needs to know what properties the desired circle has; the most direct way of determining the properties is to prove that in a figure including the circle (see Figure 15.1), PV and QV are the same length, and CV bisects angle PVQ.

The relevant proof techniques are not algorithmic, but they are somewhat routine. People experienced in the domain know that one should seek congruent triangles, and that it is appropriate to draw in the line segments CV, CP, and CQ; that one of the standard methods for proving triangles congruent (SSS, ASA, AAS, or hypotenuse-leg) should probably be used; and that this knowledge should drive the search process. We note that all of the comments made in the discussion of Table 15.1 regarding the correctness of resources and the degree of certainty with which they are held apply to relevant procedures and routine competencies—what counts is what the individual

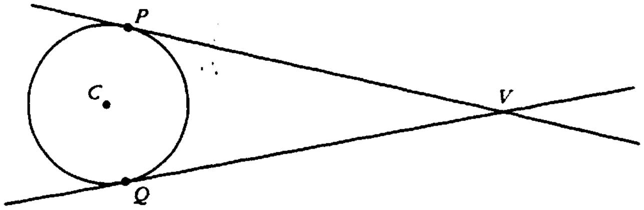


FIGURE 15–1. The Desired Configuration

holds to be true. Finally, we note the importance of understanding the rules of discourse in the domain. As noted above, Euclidean geometry is a formal game; one has to play by certain rules. For example, we can't "line up" a tangent by eye or determine the diameter of a circle by sliding a ruler along until the largest chord is obtained. While such procedures may produce the right values empirically, they are proscribed in the formal domain. People who understand this will behave very differently from those who don't.

Access to Resources (the Structure of Memory). We now turn to the issue of how the contents of memory are organized, accessed, and processed. Figure 15.2, taken from Silver (1987), provides the overarching structure for the discussion. See Norman (1970) or Anderson (1983) for general discussions.

Here, in brief, are some of the main issues brought to center stage by Figure 15.2. First is the notion that human beings are information processors, acting on the basis of their coding of stimuli experienced in the world. That is, one's experiences—visual, auditory, tactile—are registered in sensory buffers and then (if they are not ignored) converted into the forms in which they are employed in working and long-term memory. The sensory buffer (also called iconic memory, for much of its content is in the forms of images) can register a great deal of information, but hold it only briefly. Some of that information will be lost, and some will be transmitted to working memory (one can take in a broad scene perceptually, but only reproduce a small percentage of it). Speaking loosely, working or short-term memory is where "thinking gets done." Working memory receives its contents from two sources—the sensory buffer and long-term memory.

The most important aspect of working or short-term memory (STM) is its limited capacity. Pioneering research by Miller (1956) indicated that, despite the huge amount of information humans can remember in general, they can only keep and operate on about seven "chunks" of information in short-term memory. For example, a person, unless specially trained, will find it

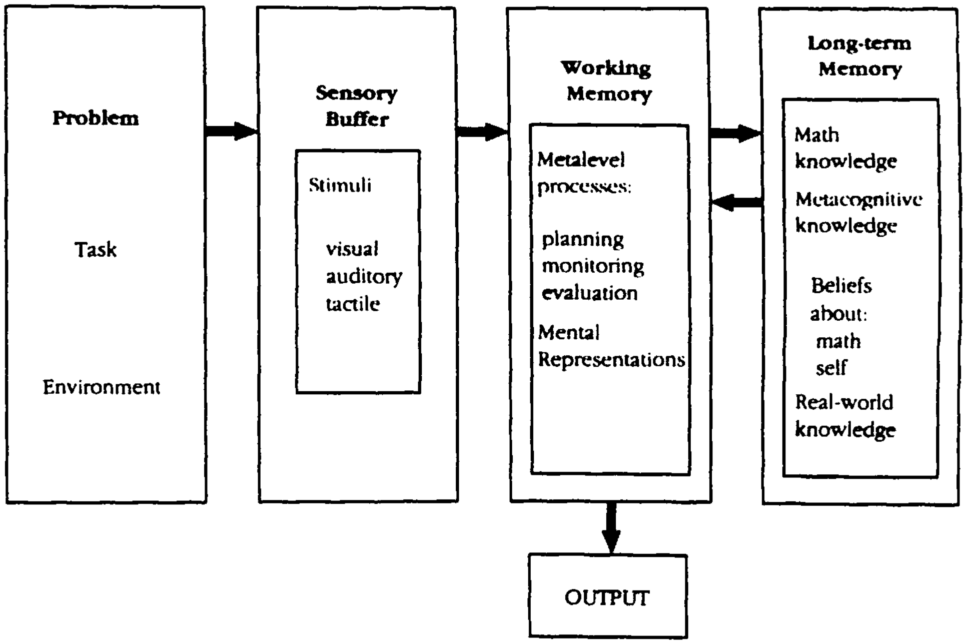


FIGURE 15–2. The Structure of Memory

nearly impossible to find the product of 637 and 829 mentally; the number of subtotals one must keep track of is too large for STM to hold. In this arithmetic example, the pieces of information in STM are relatively simple. Each of the 7 ± 2 chunks in STM can, however, be quite complex. As Simon (1980) points out:

> A chunk is any perceptual configuration (visual, auditory, or what not) that is familiar and recognizable. For those of us who know the English language, spoken and printed words are chunks... For a person educated in Japanese schools, any one of several thousand Chinese ideograms is a single chunk (and not just a complex collection of lines and squiggles), and even many pairs of such ideograms constitute single chunks. For an experienced chess player, a "fianchettoed castled Black King's position" is a chunk, describing the respective locations of six or seven of the Black pieces. (1980, p. 83)

In short, the architecture of STM imposes severe constraints on the kinds and amount of mental processing people can perform. The operation of chunking—by which one can have compound entities in the STM slots—only eases the constraints somewhat. "Working-memory load" is indeed a serious problem when people have to keep multiple ideas in mind during problem solving. The limits on working memory also suggest that for "knowledge-rich" domains [chess a generic example (see below), but mathematics certainly one as well], there are severe limitations to the amount of "thinking things out" that one can do; the contents of the knowledge base are critically important.

Long-term memory (LTM) is an individual's permanent knowledge repository. Details of its workings are still very much open to question and too fine-grained for this discussion, but a general consensus appears to be that some sort of "neural network" representation—graphs whose vertices (nodes) represent chunks in memory and whose links represent connections between those chunks—is appropriate. Independent of these architectural issues, the fundamental issues have to do with the nature of knowledge and the organization of knowledge for access (that is, to be brought into STM) and use.

Before turning to issues of organization and access, one should note a long-standing distinction between two types of knowledge, characterized by Ryle (1949) respectively, as "knowing that" and "knowing how." More modern terminology, employed by Anderson (1976), is that of "declarative" and "procedural" knowledge, respectively. The relationship between the two is not clear-cut; see Hiebert (1985) for a set of contemporary studies exploring the connections between them.

One of the domains in which the contents of memory has been best elaborated is chess. De Groot (1965) explored chess masters' competence, looking for explanations such as "spatial ability" to explain their ability to "size up" a board rapidly and play numerous simultaneous games of chess. He briefly showed experts and novices typical midgame positions and asked them to recreate the positions on nearby chess boards. The masters' performance was nearly flawless; the novices' quite poor. However, when the two groups were asked to replicate positions where pieces had been randomly placed on the chess boards, experts did no better than novices, and when they were asked to replicate positions that were *almost* standard chess positions, the masters often replicated the standard positions—the ones they expected to see. That is, the experts had "vocabularies" of chess positions, some 50,000 well-recognized configurations, which they recognized and to which they responded automatically. These vocabularies formed the base (but not the whole) of their competence.

The same, it is argued, holds in all domains, including mathematics. Depending on the knowledge architecture invoked, the knowledge chunks may be referred to as scripts (Schank & Abelson, 1977), frames (Minsky, 1975), or schemata (Hinsley, Hayes, & Simon, 1977). Nonetheless, the basic underlying notion is the same: People abstract and codify their experiences, and the codifications of those experiences shape what people see and how they behave when they encounter new situations related to the ones they have abstracted and codified. The Hinsley, Hayes, and Simon study is generic in that regard. In one part of their work, for example, they read the first few words of a problem statement to subjects and asked the subjects to categorize the problem—to say what information they expected the problem to provide and what they were likely to be asked.

> [A]fter hearing the three words "a river steamer" from a river current problem, one subject said, "It's going to be one of those river things with upstream, downstream, and still water. You are going to compare times upstream and downstream—or if the time is constant, it will be distance." ... After hearing five words of a triangle problem, one subject said, "This may be something about 'how far is he from his goal?' using the Pythagorean theorem." (1977, p. 97)

The Hinsley, Hayes, and Simon findings were summed up as follows:

1. People can categorize problems into types.
2. People can categorize problems without completely formulating them for solution. If the category is to be used to cue a schema for formulating a problem, the schema must be retrieved before formulation is complete.
3. People have a body of information about each problem type which is potentially useful in formulating problems of that type for solutions,... directing attention to important problem elements, making relevance judgments, retrieving information concerning relevant equations, etc.
4. People use category identifications to formulate problems in the course of actually solving them (Hinsley et al., 1977, p. 92).

In sum, the findings of work in domains such as chess and mathematics point strongly to the importance and influence of the knowledge base. First, it is argued that expertise in various domains depends of having access to some 50,000 chunks of knowledge in LTM. Since it takes some time (perhaps 10 seconds of rehearsal for the simplest items) for each chunk to become embedded in LTM, and longer for knowledge connections to be made, that is one reason expertise takes as long as it does to develop. Second, a lot of what appears to be strategy use is in fact reliance on well-developed knowledge chunks of the type, "in this well-recognized situation, do the following." Nonetheless, it is important not to overplay the roles of these knowledge schemata, for they do play the role of vocabulary—the basis for routine performance in familiar territory. Chess players, when playing at the limit of their own abilities, do rely automatically on their vocabularies of chess positions, but also do significant strategizing. Similarly, mathematicians have immediate access to large amounts of knowledge, but also employ a wide range of strategies when confronted with problems beyond the routine (and those, of course, are the problems mathematicians care about.) However, the straightforward suggestion that mathematics instruction focus on problem schemata does not sit well with the mathematics education community, for good reason. As noted in the historical review, IP work has tended to focus on performance but not necessarily on the basic understandings that support it. Hence, a reliance on schemata in crude form—"When you see these features in a problem, use this procedure"—may produce surface manifestations of competent behavior. However, that performance may, if not grounded in an understanding of the principles that led to the procedure, be error prone and easily forgotten. Thus, many educators would suggest caution when applying research findings from schema theory. For an elaboration of the underlying psychological ideas and the reaction from mathematics education, see the papers by Mayer (1985) and Sowder (1985).

Problem-solving Strategies (Heuristics). Discussions of problem-solving strategies in mathematics, or heuristics, must begin with Pólya. Simply put, *How to Solve It* (1945) planted the seeds of the problem-solving "movement" that flowered in the 1980s: Open the 1980 NCTM *Yearbook* (Krulik, 1980) to any page, and you are likely to find Pólya invoked, either directly or by inference in the discussion of problem-solving examples. The *Yearbook* begins by reproducing the *How to Solve It* problem-solving plan on its flyleaf and continues with numerous discussions of how to implement Pólya-like strategies in the classroom. Nor has Pólya's influence been limited to mathematics education. A cursory literature review found his work on problem solving cited in *American Political Science Review*, *Annual Review of Psychology*, *Artificial Intelligence*, *Computers and Chemistry*, *Computers and Education*, *Discourse Processes*, *Educational Leadership*, *Higher Education*, and *Human Learning*, to name just a few. Nonetheless, a close examination reveals that while his name is frequently invoked, his ideas are often trivialized. Little that goes in the name of Pólya also goes in the spirit of his work. Here we briefly follow two tracks: research exploring the efficacy of heuristics, or problem-solving strategies, and the "real world" implementation of problem-solving instruction.

MAKING HEURISTICS WORK. The scientific status of heuristic strategies such as those discussed by Pólya in *How to Solve It*—strategies in his "short dictionary of heuristics" such as (exploiting) analogy, auxiliary elements, decomposing and recombining, induction, specialization, variation, and working backwards—has been problematic, although the evidence appears to have turned in Pólya's favor over the past decade.

There is no doubt that Pólya's accounts of problem solving have face validity, in that they ring true to people with mathematical sophistication. Nonetheless, through the 1970s there was little empirical evidence to back up the sense that heuristics could be used as a means to enhance problem solving. For example, Wilson (1967) and Smith (1973) found that the heuristics that students were taught did not, despite their ostensible generality, transfer to new domains. Studies of problem-solving behaviors by Kantowski (1977), Kilpatrick (1967), and Lucas (1974) did indicate that students' use of heuristic strategies was positively correlated with performance on ability tests and on specially constructed problem-solving tests; however, the effects were relatively small. Harvey and Romberg (1980), in a compilation of dissertation studies in problem solving over the 1970s, indicated that the teaching of problem-solving strategies was "promising" but had yet to pan out. Begle had the following pessimistic assessment of the state of the art as of 1979:

> A substantial amount of effort has gone into attempts to find out what strategies students use in attempting to solve mathematical problems. ...No clear-cut directions for mathematics education are provided by the findings of these studies. In fact, there are enough indications that problem solving strategies are both problem- and student-specific often enough to suggest that finding one (or few) strategies which should be taught to all (or most) students are far too simplistic. (1979, pp. 145–146)

In other fields such as artificial intelligence, where significant attention was given to heuristic strategies, strategies of the type described by Pólya were generally ignored (see, e.g., Groner, Groner, & Bischof, 1983; Simon, 1980). Newell, in summing up Pólya's influence, states the case as follows.

> This chapter is an inquiry into the relationship of George Pólya's work on heuristic to the field of artificial intelligence (hereafter, AI). A neat phrasing of its theme would be *Pólya revered and Pólya ignored*. Pólya revered, because he is recognized in AI as the person who put heuristic back on the map of intellectual concerns. But Pólya ignored, because no one in AI has seriously built on his work....
>
> Everyone in AI, at least that part within hailing distance of problem solving and general reasoning, knows about Pólya. They take his ideas as provocative and wise. As Minsky (1961) states, "And everyone should know the work of Pólya on how to solve problems." But they also see his work as being too informal to build upon. Hunt (1975) has said "Analogical reasoning is potentially a very powerful device. In fact, Pólya [1954] devoted one entire volume of his two volume work to the discussion of the use of analogy and induction in mathematics. Unfortunately, he presents ad hoc examples but no general rules [p. 221]."

The 1980s have been kinder to heuristics à la Pólya. In short, the critique of the strategies listed in *How to Solve It* and its successors is that the characterizations of them were descriptive rather than prescriptive. That is, the characterizations allowed one to recognize the strategies when they were being used. However, Pólya's characterizations did not provide the amount of detail that would enable people who were not already familiar with the strategies to be able to implement them. Consider, for example, an ostensibly simple strategy such as "examining special cases." (This discussion is taken from pp. 288–290 of Schoenfeld [1987b, December].)

> *To better understand an unfamiliar problem, you may wish to exemplify the problem by considering various special cases. This may suggest the direction of, or perhaps the plausibility of, a solution.*

Now consider the solutions to the following three problems.

Problem 1. Determine a formula in closed form for the series

$$\sum_{k=1}^{n} \frac{k}{(k+1)!}$$

Problem 2. Let P(x) and Q(x) be polynomials whose coefficients are the same but in "backwards order":

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n, \text{ and}$$
$$Q(x) = a_n + a_{n-1}x + a_{n-2}x^2 + \ldots a_0x^n$$

What is the relationship between the roots of P(x) and Q(x)? Prove your answer.

Problem 3. Let the real numbers a_0 and a_1 be given. Define the sequence $\{a_n\}$ by

$$a_n = \tfrac{1}{2}(a_{n-2} + a_{n-1}) \text{ for each n} \geq 2.$$

Does the sequence $\{a_n\}$ converge? If so, to what value?

Details of the solutions will not be given here. However, the following observations are important. For Problem 1, the special cases that help are examining what happens when the integer parameter n takes on the values 1, 2, 3,... in sequence; this suggests a general pattern that can be confirmed by induction. Yet trying to use special cases in the same way on the second problem may get one into trouble: Looking at values $n = 1, 2, 3, \ldots$ can lead to a wild goose chase. In Problem 2, the "right" special cases of P(x) and Q(x) to look at are easily factorable polynomials. Considering $P(x) = (2x + 1)(x + 4)(3x - 2)$, for example, leads to the discovery that its "reverse" can be factored without difficulty. The roots of P and Q are easy to compare, and the result (which is best proved another way) becomes obvious. Again, the special cases that simplify the third problem are different in nature. Choosing the values $a_0 = 0$ and $a_1 = 1$ allows one to see what happens for the sequence that those two values generate. The pattern in that case suggests what happens in general and (especially if one draws the right picture!) leads to a solution of the original problem.

Each of these problems typifies a large class of problems and exemplifies a different special-cases strategy. We have

Strategy 1. When dealing with problems in which an integer parameter n plays a prominent role, it may be of use to examine values of $n = 1, 2, 3, \ldots$ in sequence, in search of a pattern.

Strategy 2. When dealing with problems that concern the roots of polynomials, it may be of use to look at easily factorable polynomials.

Strategy 3. When dealing with problems that concern sequences or series that are constructed recursively, it may be of use to try initial values of 0 and 1—if such choices don't destroy the generality of the processes under investigation.

Needless to say, these three strategies hardly exhaust "special cases." At this level of analysis—the level necessary for implementing the strategies—one could find a dozen more. This is the case for almost all of Pólya's strategies. The indications are that students can learn to use these more carefully delineated strategies (see, e.g., Schoenfeld, 1985a).

Generally speaking, studies of comparable detail have yielded similar findings. Silver (1979, 1981), for example, showed that "exploiting related problems" is much more complex than it first appears. Heller and Hungate (1985), in discussing the solution of (routine) problems in mathematics and science, indicate that attention to fine-grained detail, of the type suggested in the AI work discussed by Newell (1983), does allow for the delineation of learnable and usable problem-solving strategies. Their recommendations, derived from detailed studies of cognition are (1) make tacit processes explicit; (2) get students talking about processes; (3) provide guided practice; (4) ensure that component procedures are well learned; and (5) emphasize both qualitative understanding and specific procedures. These recommendations appear to apply to heuristic strategies as well as to the more routine techniques Heller and Hungate (1985) discuss. Similarly, Rissland's (1985) "tutorial" on AI and mathematics education points to parallels and to the kinds of advances that can be made with detailed analyses of problem-solving performance. There now exists the base knowledge for the careful, prescriptive characterization of problem-solving strategies.

"Problem Solving" in school curricula. In classroom practice, unfortunately, the rhetoric of problem solving has been seen more frequently than its substance. The following are some summary statements from the Dossey, Mullis, Lindquist, and Chambers (1988) *Mathematics Report Card*.

> Instruction in mathematics classes is characterized by teachers explaining material, working problems at the board, and having students work mathematics problems on their own—a characterization that has not changed across the eight-year period from 1978 to 1986.
>
> Considering the prevalence of research suggesting that there may be better ways for students to learn mathematics than listening to their teachers and then practicing what they have heard in rote fashion, the rarity of innovative approaches is a matter for true concern. Students need to learn to apply their newly acquired mathematics skills by involvement in investigative situations, and their responses indicate very few activities to engage in such activities. (p. 76)

According to the *Mathematics Report Card*, there is a predominance of textbooks, workbooks, and ditto sheets in mathematics classrooms; lessons are generically of the type Burkhardt (1988) calls the "exposition, examples, exercises" mode. Much the same is true of lessons that are supposedly about problem solving. In virtually all mainstream texts, "problem solving" is a separate activity and highlighted as such. Problem solving is usually included in the texts in one of two ways. First, there may be occasional "problem-solving" tasks sprinkled through the text (and delineated as such) as rewards or recreations. The implicit message contained in this format is, "You may take a breather from the real business of doing mathematics, and enjoy yourself for a while." Second, many texts contain "problem-solving" sections in which students are given drill-and-practice on simple versions of the strategies discussed in the previous section. In generic textbook fashion, students are shown a strategy (say "finding patterns" by trying values of $n = 1, 2, 3, 4$ in sequence and guessing the result in general), given practice exercises using the strategy, given homework using the strategy, and tested on the strategy. Note that when the strategies are taught this way, they are no longer heuristics in Pólya's sense; they are mere algorithms. Problem solving, in the spirit of Pólya, is learning to grapple with new and unfamiliar tasks when the relevant solution methods (even if only partly mastered) are not known. When students are drilled in solution procedures as described here, they are not developing the broad set of skills Pólya and other mathematicians who cherish mathematical thinking have in mind.

Even with good materials (and more problem sources are becoming available; see, e.g., Groves & Stacey, 1984; Mason, Burton, & Stacey, 1982; Shell Centre, 1984), the task of teaching heuristics with the goal of developing the kinds of flexible skills Pólya describes is a sometimes daunting task. As Burkhardt notes, teaching problem solving is harder for the teacher in these ways:

> Mathematically: The teachers must perceive the implications of the students' different approaches, whether they may be fruitful and, if not, what might make them so.
>
> Pedagogically: The teacher must decide when to intervene, and what suggestions will help the students while leaving the solution essentially in their hands, and carry this through for each student, or group of students, in the class.
>
> Personally: The teacher will often be in the position, unusual for mathematics teachers and uncomfortable for many, of *not knowing*; to work well without knowing all the answers requires experience, confidence, and self-awareness. (1988, p. 18)

That is, true problem solving is as demanding on the teacher as it is on the students—and far more rewarding, when achieved, than the pale imitations of it in most of today's curricula.

Self-regulation, or Monitoring and Control. Self-regulation, or monitoring and control, is one of three broad arenas encompassed under the umbrella term *metacognition*. For a broad historical review of the concept, see Brown (1987). In brief, the issue is one of resource allocation during cognitive activity and problem solving. We introduce the notion with some generic examples.

As you read some expository text, you may reach a point at which your understanding becomes fuzzy; you decide to either reread the text or stop and work out some illustrative examples to make sure you've gotten the point. In the midst of writing an article, you may notice that you've wandered from your intended outline. You may scrap the past few paragraphs and return to the original outline, or you may decide to modify it on the basis of what you've just written. Or, as you work a mathematical problem you may realize that the problem is more complex than you had thought at first. Perhaps the best thing to do is start over and make sure that you've fully understood it. Note that at this level of description, the actions in all three domains—the reading, writing, and mathematics—is much the same. In the midst of intellectual activity ("problem solving," broadly construed), you kept tabs on how well things were going. If things appeared to be proceeding well, you continued along the same path; if they appeared to be problematic, you took stock and considered other options. Monitoring and assessing progress "on line," and acting in response to the assessments of on-line progress, are the core components of self-regulation.

During the 1970s, research in at least three different domains—the developmental literature, artificial intelligence, and mathematics education—converged on self-regulation as a topic of importance. In general, the developmental literature shows that as children get older, they get better at planning for the tasks they are asked to perform and are better at making corrective judgments in response to feedback from their attempts. (Note: Such findings are generally cross-sectional, comparing the performance of groups of children at different age levels; studies rarely follow individual students or cohort groups through time.) A mainstream example of such findings is Karmiloff-Smith's (1979) study of children, ages four through nine, working on the task of constructing a railroad track. The children were given pieces of cardboard representing sections of a railroad track and told that they needed to put all of the pieces together to make a complete loop, so that the train could go around their completed track without ever leaving the track. They were rehearsed on the problem conditions until it was clear that they knew all of the constraints they had to satisfy in putting the tracks together. Typically the four- and five-year old children in the study jumped right into the task, picking up sections of the track more or less at random and lining them up in the order in which they picked them up. They showed no evidence of systematic planning for the task or in its execution. The older children in the study, ages eight and nine, engaged in a large amount of planning before engaging in the task. They sorted the track sections into piles (straight and curved track sections) and chose systematically from the piles (alternating curved and straight sections, or two straight and two curved in sequence) to build the track loops. They were, in general, more effective and efficient at getting the task done. In short, the ability and predilection to plan, act according to plan, and take on-line feedback into account in carrying out a plan seem to develop with age.

Over roughly the same time period, researchers in artificial intelligence came to recognize the necessity for "executive control" in their own work. As problem-solving programs (and expert systems) became increasingly complex, it became clear to researchers in AI that "resource management" was an issue. Solutions to the resource allocation problem varied widely, often dependent on the specifics of the domain in which planning or problem-solving was being done. Sacerdoti (1974), for example, was concerned with the time sequence in which plans are executed—an obvious concern if one tries to follow the instructions, "Put your socks and shoes on" or "Paint the ladder and paint the ceiling" in literal order. His architecture, NOAH (for Nets of Action Hierarchies), was designed to help make efficient planning decisions that would avoid execution roadblocks. NOAH's plan execution was top-down, fleshing out plans from the most general level downward, and only filling in specifics when necessary. Alternate models, corresponding to different domains were bottom-up; and still others, most notably the "Opportunistic Planning Model," or OPM, of Hayes-Roth and Hayes-Roth (1979), was heterarchical—somewhat top-down in approach, but also working at the local level when appropriate. In many ways, the work of Hayes-Roth and Hayes-Roth paralleled emerging work in mathematical problem solving. The task they gave subjects was to prioritize and plan a day's errands. Subjects were given a schematic map of a (hypothetical) city and list of tasks that should, if possible, be achieved that day. The tasks ranged from trivial and easily postponed (ordering a book) to essential (picking up medicine at the druggist). There were too many tasks to be accomplished, so the problem solver had to both prioritize the tasks and find reasonably efficient ways of sequencing and achieving them. The following paragraph summarizes those findings and stands in contrast to the generically clean and hierarchical models typifying the AI literature.

> [P]eople's planning activity is largely *opportunistic*. That is, at each point in the process, the planner's current decisions and observations suggest various opportunities for plan development. The planner's subsequent decisions follow up on selected opportunities. Sometimes these decision processes follow an orderly path and produce a neat top-down expansion.... However, some decisions and observations might suggest less orderly opportunities for plan development. For example, a decision about how to conduct initial planned activities might illuminate certain constraints on the planning of later activities and cause the planner to refocus attention on that phase of the plan. Similarly, certain low-level refinements of a previous, abstract plan might suggest an alternative abstract plan to replace the original one. (Hayes-Roth & Hayes-Roth, 1979, p. 276)

Analogous findings were accumulating in the mathematics education literature. In the early 1980s, Silver (1982), Silver, Branca, and Adams (1980), and Garofalo and Lester (1985) pointed out the usefulness of the construct for mathematics educators; Lesh (1983, 1985) focused on the instability of students' conceptualizations of problems and problem situations and of the consequences of such difficulties. Speaking loosely, all of these studies dealt with the same set of issues regarding effective and resourceful problem-solving behavior. Their results can be summed up as follows: It's not just what you know; it's how, when, and whether you use it. The focus here is on two sets of studies designed to help students develop self-regulatory skills during mathematical problem solving. The studies were chosen for discussion because of (1) the explicit focus on self-regulation in both studies, (2) the amount of time each devoted to helping students develop such skills, and (3) the detailed reflections on success and failure in each.

Schoenfeld's (1985a, 1987a) problem-solving courses at the college level have as one of their major goals the development of executive or control skills. Here is a brief summary, adapted from Schoenfeld (1989d.)

The major issues are illustrated in Figures 15.3 and 15.4. Figure 15.3 shows the graph of a problem-solving attempt by a pair of students working as a team. The students read the problem, quickly chose an approach to it, and pursued that

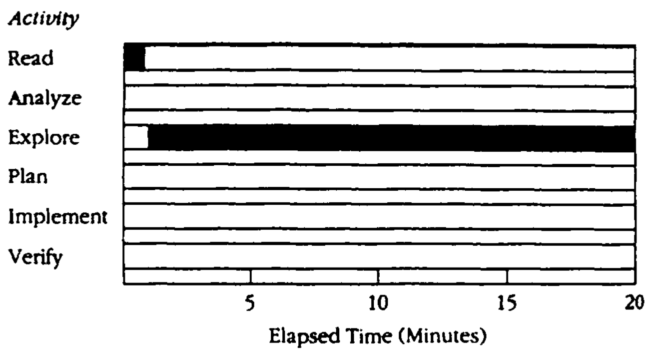


FIGURE 15–3. Time-line graph of a typical student attempt to solve a non-standard problem

approach. They kept working on it, despite clear evidence that they were not making progress, for the full 20 minutes allocated for the problem session. At the end of the 20 minutes, they were asked how that approach would have helped them to solve the original problem. They couldn't say.

The reader may not have seen this kind of behavior too often. Such behavior does not generally appear when students work routine exercises, since the problem context in that case tells the students which techniques to use. (In a unit test on quadratic equations, for example, students know that they'll be using the quadratic formula.) But when students are doing *real* problem solving, working on unfamiliar problems out of context, such behavior more reflects the norm than not. In Schoenfeld's collection of (more than a hundred) videotapes of college and high-school students working unfamiliar problems, roughly 60% of the solution attempts are of the "read, make a decision quickly, and pursue that direction come hell or high water" variety. And that first, quick, wrong decision, if not reconsidered and reversed, *guarantees* failure.

Figure 15.4, which stands in stark contrast to Figure 15.3, traces a mathematics faculty member's attempt to solve a difficult two-part problem. The first thing to note is that the mathematician spent more than half of his allotted time trying to make sense of the problem. Rather than committing himself to any one particular direction, he did a significant amount of analyzing and (structured) exploring—not spending time in unstructured exploration or moving into implementation until he was sure he was working in the right direction. Second, note

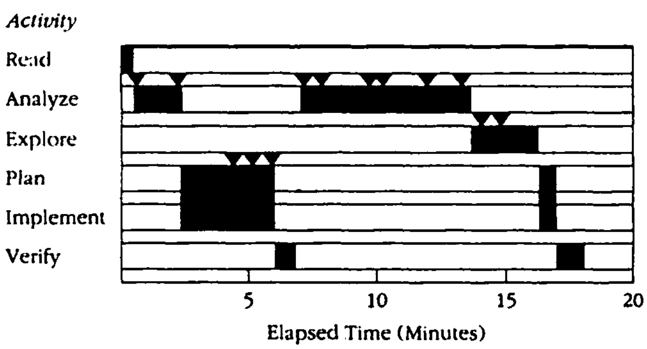


FIGURE 15–4. Time-line graph of a mathematician working a

that each of the small inverted triangles in Figure 15.4 represents an explicit comment on the state of his problem solution, for example, "Hmm. I don't know exactly where to start here" (followed by two minutes of analyzing the problem) or "OK. All I need to be able to do is [a particular technique] and I'm done" (followed by the straightforward implementation of his problem solution). It is interesting that when this faculty member began working the problem, he had fewer of the facts and procedures required to solve the problem readily accessible to him than did most of the students who were recorded working the problem. And, as he worked through the problem, the mathematician generated enough potential wild goose chases to keep an army of problem solvers busy. But he didn't get deflected by them. By monitoring his solution with care—pursuing interesting leads and abandoning paths that didn't seem to bear fruit—he managed to solve the problem, while the vast majority of students did not.

The general claim is that these two illustrations are relatively typical of adult student and "expert" behavior on unfamiliar problems. For the most part, students are unaware of or fail to use the executive skills demonstrated by the expert. However, it is the case that such skills can be learned as a result of explicit instruction that focuses on metacognitive aspects of mathematical thinking. That instruction takes the form of "coaching," with active interventions as students work on problems.

Roughly one third of the time in Schoenfeld's problem-solving classes is spent with the students working problems in small groups. The class divides into groups of three or four students and works on problems that have been distributed, while the instructor circulates through the room as "roving consultant." As he moves through the room, he reserves the right to ask the following three questions at any time:

What (exactly) are you doing? (Can you describe it precisely?)
Why are you doing it? (How does it fit into the solution?)
How does it help you? (What will you do with the outcome when you obtain it?)

He begins asking these questions early in the term. When he does so, the students are generally at a loss regarding how to answer them. With the recognition that, despite their uncomfortableness, he is going to continue asking those questions, the students begin to defend themselves against them by discussing the answers to them in advance. By the end of the term, this behavior has become habitual. (Note, however, that the better part of a semester is necessary to obtain such changes.)

The results of these interventions are best illustrated in Figure 15.5, which summarizes a pair of students' problem-solving attempt after taking the course. After reading the problem, they jumped into one solution attempt which, unfortunately, was based on an unfounded assumption. They realized this a few minutes later and decided to try something else. That choice too was a bad one, and they got involved in complicated computations that kept them occupied for $8\frac{1}{2}$ minutes. But at that point they stopped once again. One of the students said, "No, we aren't getting anything here [What we're doing isn't justi-

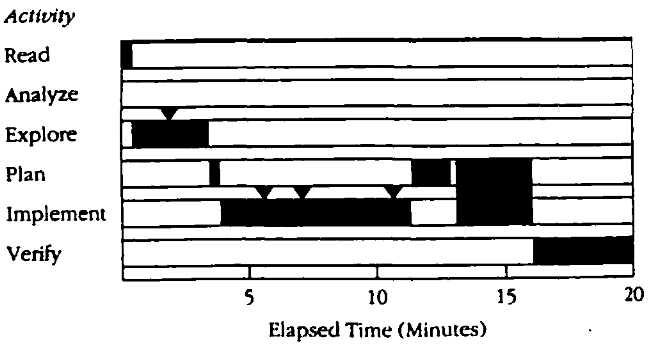


FIGURE 15-5. Time-line graph of a solution attempt after explicit training in monitoring and control.

fied]...Let's start all over and forget about this." They did, and found a solution in short order.

The students' solution is hardly expert-like in the standard sense, since they found the "right" approach quite late in the problem session. Yet in many ways their work resembled the mathematician's behavior illustrated in Figure 15.4 far more than the typical student behavior illustrated in Figure 15.3. The point here is not that the students managed to solve the problem, for to a significant degree solving nonstandard problems is a matter of luck and prior knowledge. The point is, that by virtue of good self-regulation, the students gave themselves the opportunity to solve the problem. They curtailed one possible wild goose chase shortly after beginning to work on the problem and truncated extensive computations halfway through the solution. Had they failed to do so (and they and the majority of their peers did fail to do so prior to the course), they never would have had the opportunity to pursue the correct solution they did find. In this, the students' behavior *was* expert-like. And in this, their solution was also typical of post-instruction attempts by the students. In contrast to the 60% of the "jump into a solution attempt and pursue it no matter what" attempts prior to the course, fewer than 20% of the post-instruction solution attempts were of that type. There was a concomitant increase in problem-solving success.

Lester et al. (1989, June) recently completed a major research and intervention study at the middle school-level, "designed to study the role of metacognition (i.e., the knowledge and control of cognition) in seventh graders' mathematical problem solving" (p. v). The goal of the instruction, which took place in one "regular" and one "advanced" seventh-grade mathematics class, was to foster students' metacognitive development. Ways of achieving this goal were to have the teacher (1) serve as external monitor during problem solving, (2) encourage discussion of behaviors considered important for the internalization of metacognitive skills, and (3) model good executive behavior. Table 15.2 delineates the teacher behaviors stressed in the instruction. The total instruction time focusing on metacognition in the experiment was 16.1 hours spread over 12 weeks of instruction, averaging slightly more than $\frac{1}{3}$ (35.7%) of the mathematics classroom time during the instructional period.

The instruction included both "routine" and "nonroutine" problems. An example of a routine problem designed to give students experience in translating verbal statements into mathematical expressions was as follows.

> *Laura and Beth started reading the same book on Monday. Laura read 19 pages a day and Beth read 4 pages a day. What page was Beth on when Laura was on page 133?*

The nonroutine problems used in the study included "process problems" (problems for which there is no standard algorithm for extracting or representing the given information) and problems with either superfluous or insufficient information. The instruction focused on problems amenable to particular strategies (guess-and-check, work backwards, look for patterns) and included games for whole-group activities. Assessment data and tools employed before, during, and after the instruction included written tests, clinical interviews, observations of individual and pair problem-solving sessions, and videotapes of the classroom instruction. Some of the main conclusions drawn by Lester et al. were as follows:

- There is a dynamic interaction between the mathematical concepts and processes (including metacognitive ones) used to solve problems using those concepts. That is, control processes and awareness of cognitive processes develop concurrently with an understanding of mathematical concepts.
- In order for students' problem-solving performance to improve, they must attempt to solve a variety of types of problems on a regular basis and over a prolonged period of time.
- Metacognition instruction is most effective when it takes place in a domain-specific context.
- Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.
- It is difficult for the teacher to maintain the roles of monitor, facilitator, and model in the face of classroom reality, especially when the students are having trouble with basic subject matter.
- Classroom dynamics regarding small-group activities are not as well understood as one would like, and facile assumptions that "small-group interactions are best" may not be warranted. The issue of "ideal" class configurations for problem-solving lessons needs more thought and experimentation.
- Assessment practices must reward and encourage the kinds of behaviors we wish students to demonstrate (1989, pp. 88–95).

Briefly, the findings discussed in this section are that developing self-regulatory skills in complex subject-matter domains is difficult and often involves behavior modification—"unlearning" inappropriate control behaviors developed through prior instruction. Such change can be catalyzed, but it requires a long period of time, with sustained attention to both cognitive and metacognitive processes. The task of creating the "right" instructional context, and providing the appropriate kinds of modeling and guidance, is challenging and subtle for

TABLE 15–2. Teaching Actions for Problem-Solving

Teaching Action	Purpose
BEFORE	
1. Read the problem—discuss words or phrases students may not understand	Illustrate the importance of reading carefully; focus on special vocabulary
2. Use whole-class discussion to focus on importance of understanding the problem	Focus on important data, clarification process
3. (Optional) Whole-class discussion of possible strategies to solve a problem	Elicit ideas for possible ways to solve the problem
DURING	
4. Observe and question students to determine where they are	Diagnose strengths and weaknesses
5. Provide hints as needed	Help students past blockages
6. Provide problem extensions as needed	Challenge early finishers to generalize
7. Require students who obtain a solution to "answer the question"	Require students to look over their work and make sure it makes sense
AFTER	
8. Show and discuss solutions	Show and name different strategies
9. Relate to previously solved problems or have students solve extensions	Demonstrate general applicability of problem solving strategies
10. Discuss special features, e.g. pictures	Show how features may influence approach

(Adapted from Lester et al., 1989, p. 26)

the teacher. The two studies cited (Lester et al., 1989; Schoenfeld, 1989d) point to some effective teacher behaviors and classroom practices that foster the development of self-regulatory skills. However, these represent only a beginning. They document the teaching efforts of established researchers who have the luxury to reflect on such issues and prepare instruction devoted to them. Making the move from such "existence proofs" (problematic as they are) to standard classrooms will require a substantial amount of conceptualizing and pedagogical engineering.

Beliefs and Affects

Once upon a time there was a sharply delineated distinction between the cognitive and affective domains, as reflected in the two volumes of Bloom's (1956) *Taxonomy of Educational Objectives*. Concepts such as mathematics anxiety, for example, clearly resided in the affective domain and were measured by questionnaires dealing with how the individual feels about mathematics (see, e.g., Suinn, Edie, Nicoletti, & Spinelli, 1972). Concepts such as mathematics achievement and problem solving resided within the cognitive domain and were assessed by tests focusing on subject-matter knowledge alone. As our vision gets clearer, however, the boundaries between those two domains become increasingly blurred.

Given the space constraints, to review the relevant literature or even try to give a sense of it would be an impossibility. Fortunately, one can point to McLeod, Chapter 23, this volume and to books such as McLeod and Adams's (1989) *Affect and Mathematical Problem Solving: A New Perspective* as authoritative starting points for a discussion of affect. Beliefs—to be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior—will receive a telegraphic discussion. The discussion will take place in three parts: student beliefs, teacher beliefs, and general societal beliefs about doing mathematics. There is a fairly extensive literature on the first, a moderate but growing literature on the second, and a small literature on the third. Hence, length of discussion does not correlate with the size of the literature base.

Student Beliefs. As an introduction to the topic, we recall Lampert's commentary:

Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly (Ball, 1988; Schoenfeld, 1985b; Stodolsky, 1985). These cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing. (1990, p. 31)

An extension of Lampert's list, including other student beliefs delineated in the sources she cites, is given in Table 15.3.

The basic arguments regarding student beliefs were made in the first part of this chapter. As an illustration, consider the genesis and consequences of one belief regarding the amount of time students think is appropriate to spend working mathematics problems. The data come from year-long observations of high-school geometry classes.

Over the period of a full school year, none of the students in any of the dozen classes we observed worked mathematical tasks that could seriously be called problems. What the students worked were exercises: tasks designed to indicate mastery of relatively small chunks of subject matter, and to be completed in a short amount of time. In a typical five-day sequence, for example, students were given homework assignments that consisted of 28, 45, 18, 27, and 30 "problems" respectively. ...[A particular] teacher's practice was to have students present solutions to as many of the homework problems as possible at the board. Given the length of his assignments, that means that he expected the students to be able to work twenty or more "problems" in a fifty-four minute class period. Indeed, the unit test on locus and construction problems (a uniform exam in Math 10 classes at the school) contained twenty-five problems—giving students an average two minutes and ten seconds to work each problem. The teacher's advice to the students summed things up in a nutshell: "You'll have to know all your constructions cold *so you don't spend a lot of time thinking about them.*" [emphasis added]...

In sum, students who have finished a full twelve years of mathematics have worked thousands upon thousands of "problems"—virtually none of which were expected to take the students more than a few minutes to complete. The presumption underlying the assignments was as follows: If you understand the material, you can work the exercises. If you can't work the exercises within a reasonable amount of time, then you don't understand the material. That's a sign that you should seek help.

TABLE 15-3. Typical Student Beliefs about the Nature of Mathematics

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class.
- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically and without understanding.
- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learned in school has little or nothing to do with the real world.
- Formal proof is irrelevant to processes of discovery or invention.

Whether or not the message is intended, students get it. One of the open-ended items on our questionnaire, administered to students in twelve high school mathematics classes in grades 9 through 12, read as follows: "If you understand the material, how long should it take to answer a typical homework problem? What is a reasonable amount of time to work on a problem before you know it's impossible?" Means for the two parts of the question were 2.2 minutes (n = 221) and 11.7 minutes (n = 227), respectively. (Schoenfeld, Spring 1988, pp. 159–160)

Unfortunately, this belief has a serious behavioral corollary. Students with the belief will give up on a problem after a few minutes of unsuccessful attempts, even though they might have solved it had they persevered.

There are parallel arguments regarding the genesis and consequences of the each of the beliefs listed in Table 15.3. Recall, for example, the discussion of the artificial nature of Milne's mental arithmetic problems in the first part of this chapter. It was argued that, after extended experience with "cover stories" for problems that are essentially algorithmic exercises, students come to ignore the cover stories and focus on the "bottom line": performing the algorithm and writing down the answer. That kind of behavior produced an astonishing and widely quoted result in the third National Assessment of Educational Progress (NAEP, 1983), when a plurality of students who performed the correct numerical procedure on a problem ignored the cover story for the problem and wrote that the number of buses required for a given task was "31 remainder 12." In short:

1. Students abstract their beliefs about formal mathematics—their sense of their discipline—in large measure from their experiences in the classroom.
2. Students' beliefs shape their behavior in ways that have extraordinarily powerful (and often negative) consequences.

Teacher Beliefs. Belief structures are important not only for students, but for teachers as well. Simply put, a teacher's sense of the mathematical enterprise determines the nature of the classroom environment that the teacher creates. That environment, in turn, shapes students' beliefs about the nature of mathematics. We briefly cite two studies that provide clear documentation of this point. Cooney (1985) discussed the classroom behavior of a beginning teacher who professed a belief in "problem solving." At bottom, however, this teacher felt that giving students "fun" or nonstandard problems to work—that teacher's conception of problem solving—was, although recreational and motivational, ultimately subordinate to the goal of having students master the subject matter to be covered. Under the pressures of content coverage, the teacher sacrificed (essentially superficial) problem-solving goals for the more immediate goals of drilling students on the things they would be held accountable for.

Thompson (1985) presents two case studies demonstrating the ways that teacher beliefs play out in the classroom. One of her informants was named Jeanne.

Jeanne's remarks revealed a view of the content of mathematics as fixed and predetermined, as dictated by the physical world. At no time during

either the lessons [Thompson observed] or the interviews did she allude to the generative processes of mathematics. It seemed apparent that she regarded mathematics as a finished product to be assimilated....
Jeanne's conception of mathematics teaching can be characterized in terms of her view of her role in teaching the subject matter and the students' role in learning it. Those were, in gross terms, that she was to disseminate information, and that her students were to receive it. (p. 286)

These beliefs played out in Jeanne's instruction. The teacher's task, as she saw it, was to present the lesson planned, without digressions or inefficient changes. Her students experienced the kind of rigid instruction that leads to the development of some of the student beliefs described above.

Thompson's second informant was named Kay. Among Kay's beliefs about mathematics and pedagogy were the following:

- Mathematics is more a subject of ideas and mental processes than a subject of facts.
- Mathematics can be best understood by rediscovering its ideas.
- Discovery and verification are essential processes in mathematics.
- The main objective of the study of mathematics is to develop reasoning skills that are necessary for solving problems.
- The teacher must create and maintain an open and informal classroom atmosphere to insure the students' freedom to ask questions and explore their ideas.
- The teacher should encourage students to guess and conjecture and should allow them to reason things on their own rather than show them how to reach a solution or an answer.
- The teacher should appeal to students' intuition and experiences when presenting the material in order to make it meaningful (pp. 288–290).

Kay's pedagogy was consistent with her beliefs and resulted in a classroom atmosphere that was at least potentially supportive of the development of her students' problem-solving abilities.

One may ask, of course, where teachers obtain their notions regarding the nature of mathematics and of the appropriate pedagogy for mathematics instruction. Not surprisingly, Thompson notes: "There is research evidence that teachers' conceptions and practices, particularly those of beginning teachers, are largely influenced by their schooling experience prior to entering methods of teaching courses" (p. 292). Hence, teacher beliefs tend to come home to roost in successive generations of teachers, in what may for the most part be a vicious pedagogical/epistemological circle.

Societal Beliefs. Stigler & Perry (1989) report on a series of cross-cultural studies that serve to highlight some of the societal beliefs in the United States, Japan, and China regarding mathematics.

There are large cultural differences in the beliefs held by parents, teachers, and children about the nature of mathematics learning. These beliefs can be organized into three broad categories: beliefs about what is *possible*, (i.e., what children are able to learn about mathematics at different ages); beliefs about what is *desirable* (i.e., what children should learn); and beliefs about what is the best *method* for teaching mathematics (i.e., how children should be taught). (p. 196)

Regarding what is possible, the studies indicate that people in the United States are much more likely than the Japanese to believe that innate ability (as opposed to effort) underlies children's success in mathematics. Such beliefs play out in important ways. First, parents and students who believe "either you have it or you don't" are much less likely to encourage students to work hard on mathematics than those who believe "you can do it if you try." Second, our nation's textbooks reflect our uniformly low expectations of students. "U.S. elementary textbooks introduce large numbers at a slower pace than do Japanese, Chinese, or Soviet textbooks, and delay the introduction of regrouping in addition and subtraction considerably longer than do books in other countries" (Stigler & Perry, 1989, p. 196). Regarding what is desirable, the studies indicate that, despite the international comparison studies, parents in the United States believe that reading, not mathematics, needs more emphasis in the curriculum. And finally, on methods:

Those in the U.S., particularly with respect to mathematics, tend to assume that understanding is equivalent to sudden insight. With mathematics, one often hears teachers tell children that they "either know it or they don't," implying that mathematics problems can either be solved quickly or not at all.... In Japan and China, understanding is conceived of as a more gradual process, where the more one struggles the more one comes to understand. Perhaps for this reason, one sees teachers in Japan and China pose more difficult problems, sometimes so difficult that the children will probably not be able to solve them within a single class period. (Stigler & Perry, 1989, p. 197)

In sum, whether acknowledged or not, whether conscious or not, beliefs shape mathematical behavior. Beliefs are abstracted from one's experiences and from the culture in which one is embedded. This leads to the consideration of mathematical practice.

Practices

As an introduction to this section, we recall Resnick's comments regarding mathematics instruction:

Becoming a good mathematical problem solver—becoming a good thinker in any domain—may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge. If this is so, we may do well to conceive of mathematics education less as an instructional process (in the traditional sense of teaching specific, well-defined skills or items of knowledge), than as a socialization process. (1989, p. 58)

The preceding section on beliefs and affects described some of the unfortunate consequences of entering the wrong kind of mathematical practice—the practice of "school mathematics." Here we examine some positive examples. These classroom environments, designed to reflect selected aspects of the mathematical community, have students interact (with each other and the mathematics) in ways that promote mathematical thinking. We take them in increasing grade order.

Lampert (1990) explicitly invokes a Pólya-Lakatosian epistemological backdrop for her fifth-grade lessons on exponentiation, deriving pedagogical practice from that epistemological stance. She describes

a research and development project in teaching designed to examine whether and how it might be possible to bring the practice of knowing mathematics in school closer to what it means to know mathematics within the discipline by deliberately altering the roles and responsibilities of teacher and students in classroom discourse.... A [representative] case of teaching and learning about exponents derived from lessons taught in the project is described and interpreted from mathematical, pedagogical, and sociolinguistic perspectives. To change the meaning of knowing and learning in school, the teacher initiated and supported social interactions appropriate to making mathematical arguments in response to students' conjectures. The activities in which students engaged as they asserted and examined hypotheses about the mathematical structures that underlie their solutions to problems are contrasted with the conventional activities that characterize school mathematics. (p. 1)

Lampert describes a series of lessons on exponents in which students first found patterns of the last digits in the squares of natural numbers and then explored the last digits of large numbers (e.g., What is the last digit of 7^5?). In the process of classroom discussion, students found patterns, made definitions, reasoned about their claims, and ultimately defended their claims on mathematical grounds. At one point, for example, a student named Sam asserted flatly that the last digit of 7^5 is a 7, while others claimed that it was 1 or 9.

[Lampert] said: "You must have a proof in mind, Sam, to be so sure," and then [she] asked, "Arthur, why do you think it's a 1?"

The students attempted to resolve the problem of having more than one conjecture about what the last digit in seven to the fifth power might be. [The discussion] was a zig-zag between proofs that the last digit must be 7 and refutations of Arthur's and Sarah's alternative conjectures. The discussion ranged between observations of particular answers and generalizations about how exponents—and numbers more generally—work. Students examined their own assumptions and those of their classmates. [Lampert] assumed the role of manager of the discussion and sometimes participated in the argument, refuting a student's assertion. (p. 47)

At the end of the lesson, in which the class explored simple ways of looking at the last digits of 7^8 and 7^{16},

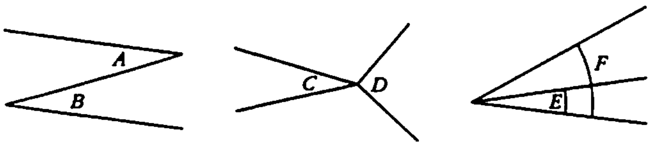
some students were verging on declaring an important law of exponents: $(n^a)(n^b) = n^{a+b}$, which they would articulate more fully, and prove the legitimacy of, in the next few classes. They were also beginning to develop a modular arithmetic of "last digits" to go with different base numbers, leading them into further generalizations about the properties of exponents. (pp. 54-55)

Note that Lampert did not "reveal truth," but entered the dialogue as a knowledgeable participant—a representative of the mathematical community who was not an all-knowing authority but rather one who could ask pointed questions to help students arrive at the correct mathematical judgments. Her pedagogical practice, in deflecting undue authority from the teacher, placed the burden of mathematical judgment (with constraints) on the shoulders of the students.

Balacheff (1987) exploits social interactions in a different way, but with similar epistemological goals. He describes a series of lessons for seventh graders, concerned with the theorem that "the sum of the angles of a triangle is 180°." The lessons begin with the class divided into small groups. Each group is given a worksheet with a copy of the same triangle and asked to compute the sum of its angles. The groups then report their answers, which vary widely—often from as little as 100° to as much as 300°! Since the students know they had all measured the same triangle, this causes a tension that must be resolved; they work on it until all students agree on a value. Balacheff then hands out a different triangle to each group, and has the group conjecture the sum of the angles before measuring it. Groups compare and contrast their results, and repeat the process with each other's triangles. This causes conflicts within and across groups, and the discussions that result in the resolutions of those conflicts make the relevant mathematical issues salient and meaningful to the students, so that they are intellectually prepared for the theoretical discussions (of a similar dialectical nature) that follow.

In a classic study that is strikingly contemporary in its spirit, Fawcett (1938) describes a two-year course in plane geometry he taught at the Ohio State University Laboratory School in the 1930s. Fawcett's goals were that students develop a good understanding of the subject matter of geometry, the right epistemological sense about the mathematics, and a sense of the applicability of the reasoning procedures that they had learned in geometry to situations outside the mathematics classroom. In order for this to happen, he believed, (1) the students had to engage in doing mathematics in a way consistent with his mathematical epistemology, (2) the connections between mathematical reasoning in the formal context of the classroom and mathematical reasoning outside it would have to be made explicit, and (3) the students would need to reflect both on their doing of mathematics and on the connections between the reasoning in both contexts.

For example, the issue of definition is important in mathematics. Fawcett pointed out that definitions have consequences: In his school, for example, there was an award for the "best teacher." Many students favored the librarian—but was the librarian a teacher? He also used sports as an analogy. In baseball, for example, there might be varying definitions of "foul ball" (Is a fly ball that hits the foul pole fair or foul?), but once the rules are set, the game can be played with consistency. After such discussions, Fawcett notes "no difficulty was met in leading the pupils to realize that these rules were nothing more than agreements which a group of interested people had made and that they implied certain conclusions" (p. 33). In the mathematical domain, he had his students debate the nature and usefulness of various definitions. Rather than provide the definition of "adjacent angle," for example, he asked the class to propose and defend various definitions. The first was "angles that share a common side," which was ruled out by Figure 15.6(*a*). A second suggestion, "angles that share a common vertex," was ruled out by Figure 15.6(*b*). "Angles that share a common side and a common vertex" had a good deal



a. two angles that share a common side

b. two angles that share a common vertex

c. two angles that share a common side and a common vertex

FIGURE 15–6. Student definitions of "adjacent angles" discussed in Fawcett (1938).

of support until it was ruled out by Figure 15.6(*c*). Finally the class agreed upon a mathematically correct definition.

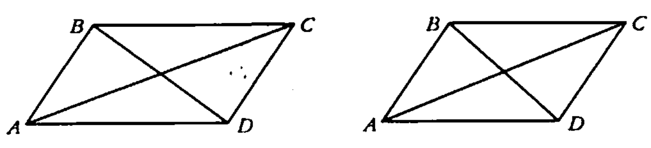
To recall a statement on the nature of mathematical *doing* by Pólya, "To a mathematician who is active in research, mathematics may appear sometimes as a guessing game; you have to guess a mathematical theorem before you prove it, you have to guess the idea of a proof before you carry through all the details" (1954, p. 158). Fawcett's class was engineered along these lines. He never gave assignments of the following form:

> *Prove that the diagonals of a parallelogram bisect each other but are not necessarily mutually perpendicular; also prove that the diagonals of a rhombus are mutually perpendicular.*

Instead, he would pose the problems in the following form:

1. Consider the parallelogram ABCD in Figure 15.7(*a*), with diagonals AC and BD. State all the properties of the figure that you are willing to accept. Then, give a complete argument justifying why you believe your assertions to be correct.
2. Suppose you also assume that AB = BC, so that the quadrilateral ABCD is a rhombus [Figure 15.7(*b*)]. State all the additional properties of the figure that you are willing to accept. Then, give a complete argument justifying why you believe your additional assertions to be correct.

Needless to say, different students had different opinions regarding what they would accept as properties of the figures. Fawcett had students supporting the different positions argue their conclusions—that is, a claim about a property of either figure had to be defended mathematically. The class (with Fawcett serving as an "especially knowledgeable member," but not as sole authority) served as "jury." Class discussions included not



a. ABCD is a parallelogram. What do you think must be true?

b. ABCD is a rhombus. What else do you think must be true?

FIGURE 15–7. Problems posed in the style of Fawcett (1938)

only what was right and wrong (i.e., Does a figure have a given property?), but also reflections on the nature of argumentation itself: Are inductive proofs always valid? Are converses always true? and so on. In short, Fawcett's students were acting like mathematicians, at the limits of their own community's (the classroom's) knowledge.

We continue with two examples at the college level. Alibert and his colleagues (1988) have developed a calculus course at Grenoble based on the following principles:

1. Coming to grips with uncertainty is a major part of the learning process.
2. A major role of proofs (the product of "scientific debate") is to convince first oneself, and then others, of the truth of a conjecture.
3. Mathematical tools can evolve meaningfully from the solution of complex problems, often taken from the physical sciences.
4. Students should be induced to reflect on their own thought processes.

Their course, based on these premises, introduces major mathematics topics with significant problems from the physical sciences (for example, the Riemann integral is introduced and motivated by a problem asking students to determine the gravitational attraction exerted by a stick on a marble). While in typical calculus classes the historical example would soon be abandoned and the subject matter would be presented in cut-and-dried fashion, the Grenoble course is true to its principles. The class, in a debate resembling that discussed in the examples from Lampert and Fawcett, formulates the mathematical problem and resolves it (in the sense of the term used by Mason et al., 1982) by a discussion in which ideas spring from the class and are nurtured by the instructor, who plays a facilitating and critical, rather than show-and-tell, role.

According to Alibert, experiences of this type result in the students' coming to grips with some fundamental mathematical notions. After the course, notes Alibert, "their conceptions of mathematics are interesting—and important for their learning. A large majority of the students answer the . . . question 'What does mathematics mean to you?' at an epistemological level; their school epistemology has almost disappeared" (p. 35).

Finally, Schoenfeld's problem-solving courses at the college level have many of the same attributes. As in Fawcett's case, no problems are posed in the "prove that" format; all are "What do you think is true, and why?" questions. Schoenfeld (forthcoming) explicitly deflects teacher authority to the student community, both in withholding his own understandings of problem solutions (many problems the class works on for days or weeks are problems for which he could present a 10-minute lecture solution) and developing in the class the critical sense of mathematical argumentation that leads it, as a community, to accept or reject on appropriate mathematical grounds the proposals made by class members.

For example, in a discussion of the Pythagorean theorem Schoenfeld (1990b) posed the problem of finding all solutions in integers to the equation $a^2 + b^2 = c^2$. There is a known solution, which he did not present. The class made a series of observations, among them:

1. Multiples of known solutions (e.g., the {6, 8, 10} right triangle as a multiple of the {3, 4, 5}) are easy to obtain, but of no real interest. The class would focus on triangles whose sides were relatively prime.
2. The class observed, conjectured, and proved that in a relatively prime solution, the value of c is always odd.
3. Students observed that in all the cases of relatively prime solutions they knew—{3, 4, 5}, {5, 12, 13}, {7, 24, 25}, {8, 15, 17}, {12, 35, 37}—the larger leg (b) and the hypotenuse (c) differed by either 1 or 2. They conjectured that there are infinitely many triples in which b and c differ by 1 and by 2, and no others.
4. They proved that there are infinitely many solutions where b and c differ by 1 and where b and c differ by 2; they proved there are no solutions where b and c differ by 3. At that point a student asked if, should the pattern continue (i.e., if they could prove their conjecture), they would have a publishable theorem.

Of course, the answer to the student's question was no. First, the conjecture was wrong: There is, for example, the {20, 21, 29} triple. Second, the definitive result—all Pythagorean triples are of the form $\{M^2 - N^2, 2MN, M^2 + N^2\}$—is well known and long established within the mathematical community. But to dismiss the students' results is to do them a grave injustice. In fact, all three of the results proved by the students in number 4 above were new to the instructor. The students were *doing* mathematics, at the frontiers of their community's knowledge.

In all of the examples discussed in this section, classroom environments were designed to be consonant with the instructors' epistemological sense of mathematics as an ongoing, dynamic discipline of sense making through the dialectic of conjecture and argumentation. In all, the authors provide some anecdotal and some empirically "objective" documentations of success. Yet, the existence of these positive cases raises far more questions that it answers. The issues raised here, and in general by the research discussed in this chapter, are the focus of discussion in the next section.

ISSUES

We conclude with an assessment of the state of the art in each of the areas discussed in this paper, pointing to both theoretical and practical issues that need attention and clarification. Caveat lector: The comments made here reflect the opinions of the author and may be shared to various degrees by the research community at large.

This chapter has focused on an emerging conceptualization of mathematical thinking based on an alternative epistemology in which the traditional conception of domain knowledge plays an altered and diminished role, even when it is expanded to include problem-solving strategies. In this emerging view, metacognition, belief, and mathematical practices are considered critical aspects of thinking mathematically. But there is more. The person who thinks mathematically has a particular way of seeing the world, of representing it, of analyzing it. Only within that overarching context do the pieces—the knowledge base, strategies, control, beliefs, and practices—fit together coherently. We begin the discussion with comments on what it might mean for the pieces to fit together.

A useful idea to help analyze and understand complex systems is that of a nearly decomposable system. The idea is that one can make progress in understanding a large and complex system by carefully abstracting from it subsystems for analysis and then combining the analyses of the subsystems into an analysis of the whole. The study of human physiology provides a familiar example. Significant progress in our understanding of physiology has been made by conducting analyses of the circulatory system, the respiratory system, the digestive system, and so on. Such analyses yield tremendous insights and help move us forward in understanding human physiology as a whole. However, insights at the subsystem level alone are insufficient: Interactions among the subsystems must be considered, and the whole is obviously much more than the sum of its parts.

One can argue, convincingly I think, that the categories in the framework identified and discussed in the second part of this chapter provide a coherent and relatively comprehensive near decomposition of mathematical thinking (or at least, mathematical behavior). The individual categories cohere, and within them (to varying degrees of success) research has produced some ideas regarding underlying mechanisms. But the research community understands little about the interactions among the categories, and less about how they come to cohere—in particular how an individual's learning in those categories fits together to give the individual a sense of the mathematical enterprise, his or her "mathematical point of view." My own bias is that the key to this problem lies in the study of enculturation, of entry into the mathematical community. For the most part, people develop their sense of any serious endeavor—be it their religious beliefs, their attitude toward music, their identities as professionals or workers, their sense of themselves as readers (or nonreaders), or their sense of mathematics—from interactions with others. And if we are to understand how people develop their mathematical perspectives, we must look at the issue in terms of the mathematical communities in which students live and the practices that underlie those communities. The role of interactions with others will be central in understanding learning, whether it be understanding how individuals come to grips with the specifics of the domain (Moschkovich, 1989; Newman, Griffin, & Cole, 1989; Schoenfeld, Smith, & Arcavi, in press) or more broad issues about developing perspectives and values (Lave & Wenger, 1989; Schoenfeld, 1989c, forthcoming). This theme will be explored a bit more in the section on practices. We now proceed with a discussion of issues related to research, instruction, and assessment.

Fundamental issues remain unaddressed or unresolved in the general area of problem solving and in each of the particular areas addressed in the second part of this chapter. To begin, the field needs much greater clarity of the meanings of the term "problem solving." The term has served as an umbrella under which radically different types of research have been conducted. At minimum there should be a de facto requirement (now the exception rather than the rule) that every study or discussion of problem solving be accompanied by an operational definition of the term and examples of what the author means—whether it be working the exercises at the end of the chapter, scoring well on the Putnam exam, or "developing a mathematical point of view and the tools to go with it" as discussed in this chapter. Although one is loath to make recommendations that may result in jargon proliferation, it seems that the time is overdue for researchers to form some consensus on definitions about various aspects of problem solving. Great confusion arises when the same term refers to a multitude of sometimes contradictory and typically underspecified behaviors.

Along the same general lines, much greater clarity is necessary with regard to research methods. It is generally accepted that all research methodologies (1) address only particular aspects of problem solving behavior, leaving others unaddressed; (2) cast some behaviors into high relief, allowing for a close analysis of those; and (3) either obscure or distort other behaviors. The researchers' tool kit is expanding from the collection of mostly statistical and experimental techniques largely employed through the 1970s (comparison studies, regression analyses, and so on) to the broad range of clinical, protocol analysis, simulation, and computer modeling methods used today. Such methods are often ill- or inappropriately used. Those we understand well should, perhaps, come with "user's guides" of the following type: "This method is suited for explorations of A, B, and C, with the following caveats; or, It has not proven reliable for explorations of D, E, and F." Here is one example, as a case in point:

> The protocol parsing scheme, that produced Figures 15.3, 15.4, and 15.5, analyzed protocols gathered in non-interventive problem solving sessions, is appropriate for documenting the presence or absence of executive decisions in problem solving, and demonstrating the consequences of those executive decisions. However, it is likely to be useful only on problems of Webster's type 2—"perplexing or difficult" problems, in which individuals must make difficult choices about resource allocation. (Control behavior is unlikely to be necessary or relevant when individuals are working routine or algorithmic exercises.) Moreover, the method reveals little or nothing about the mechanisms underlying successful or unsuccessful monitoring and assessment. More interventive methods will almost certainly be necessary to probe, on the spot, why individuals did or did not pursue particular options during problem solving. These, of course, will disturb the flow of problem solutions; hence the parsing method will no longer be appropriate for analyzing those protocols.

Indeed, a contemporary guide to research methods would be a useful tool for the field.

With regard to resources (domain knowledge), the two main issues that require attention are (1) finding adequate descriptions and representations of cognitive structures, and (2) elaborating the dynamic interaction between resources and other aspects of problem-solving behavior as people engage in mathematics. Over the past decade, researchers have developed some careful and fine-grained representations of mathematical structures, but the field still has a way to go before there is a strong congruence between the ways we describe knowledge structures and our sense of how such structures work phenomenologically. And we still lack a good sense of how the pieces fit together. How do resources interact with strategies, control, beliefs, and practices?

Much of the theoretical work with regard to problem-solving strategies has already been done; the remaining issues are more on the practical and implementational levels. The spadework for the elaboration of problem-solving strategies exists, in that there is a blueprint for elaborating strategies. It has been shown that problem-solving strategies can be described, in detail, at a level that is learnable. Following up on such studies, we now need carefully controlled data on the nature and amount of training and over what kinds of problems, that result in the acquisition of particular strategies (and how far strategy acquisition transfers). That is a demanding task, but not a theoretically difficult one.

We have made far less progress with regard to control. The importance of the idea has been identified and some methodological tools have been developed for charting control behaviors during problem solving. Moreover, research indicates that students (at least at the advanced secondary and college level) can be taught to develop productive control behaviors, although only in extended instruction that, in effect, amounts to behavior modification. However, there remain some fundamental issues to be resolved.

The first issue is mechanism. We lack an adequate characterization of control. That is, we do not have good theoretical models of what control is, and how it works. We do not know, for example, whether control is domain-independent or domain-dependent; nor do we know what the mechanisms might be for tying control decisions to domain knowledge. The second issue is development. We know that in some domains, children can demonstrate astonishingly subtle self-regulatory behaviors, in social situations, for example, where they pick up behavioral and conversational cues regarding whether and how to pursue particular topics of conversation with their parents. How and when do children develop such skills in the social domain? How and when do they develop (or fail to develop) the analogous skills in the domain of mathematics? Are the similarities merely apparent, or do they have a common base in some way? We have barely a clue regarding the answers to these questions.

The arena of beliefs and affects is reemerging as a focus of research, and it needs concentrated attention. It is basically underconceptualized, and it stands in need of new methodologies and new explanatory frames. The older measurement tools and concepts found in the affective literature are simply inadequate; they are not at a level of mechanism and most often tell us that *something* happens without offering good suggestions as to how and why. Recent work on beliefs points to issues of importance that straddle the cognitive and affective domains, but much of that work is still at the "telling good stories" level rather than the level of providing solid explanations. Despite some theoretical advances in recent years and increasing interest in the topic, we are still a long way from a unified perspective that allows for the meaningful integration of cognition and affect or, if such unification is not possible, from understanding why it is not.

Issues regarding practices and the means by which they are learned—enculturation—may be even more problematic. Here, in what may ultimately turn out to be one of the most important arenas for understanding the development of mathematical thinking, we seem to know the least. The importance of enculturation has now been recognized, but the best we can offer thus far in explication of it is a small number of well-described case studies. Those studies, however, give only the barest hints at underlying mechanisms. On the one hand, the tools available to cognitivists have yet to encompass the kinds of social issues clearly relevant for studies of enculturation—such as how one picks up the biases and perspectives common to members of a particular subculture. On the other hand, extant theoretical means for discussing phenomena such as enculturation do not yet operate at the detailed level that results in productive discussions of what people learn (e.g., about mathematics) and why. There are hints regarding theoretical means for looking at the issue, such as Lave and Wenger's (1989) concept of "legitimate peripheral participation." Roughly, the idea is that by sitting on the fringe of a community, one gets a sense of the enterprise; as one interacts with members of the community and becomes more deeply embedded in it, one learns its language and picks up its perspectives as well. It remains to be seen, however, how such means will be developed and whether they will be up to the task.

Turning to practical issues, one notes that there is a host of unsolved and largely unaddressed questions dealing with instruction and assessment. It appears that as a nation we will be moving rapidly in the direction of new curricula, some of them very much along the lines suggested in this chapter. At the national level, *Everybody Counts* (National Research Council, 1989) represents the Mathematical Sciences Education Board's attempt to focus discussion on issues of mathematics education. *Everybody Counts* makes the case quite clearly that a perpetuation of the status quo is a recipe for disaster, and it calls for sweeping changes. The NCTM *Standards* (National Council of Teachers of Mathematics, 1989) reflects an emerging national consensus that all students should study a common core of material for (a minimum of) three years in secondary school. *Reshaping School Mathematics* (National Research Council, 1990a) supports the notion of a three-year common core and provides a philosophical rationale for a curriculum focusing on developing students' mathematical power. With such national statements as a backdrop, some states are moving rapidly toward the implementation of such curricula. In California, for example, the 1985 *Mathematics Framework* (California State Department of Education, 1985) claimed that "mathematical power, which involves the ability to discern mathematical relationships, reason logically, and use mathematical techniques effectively, must be the central concern of mathematics education" (p. 1). Its classroom recommendations for teachers are as follows:

- Model problem-solving behavior whenever possible, exploring and experimenting along with students.
- Create a classroom atmosphere in which all students feel comfortable trying out ideas.
- Invite students to explain their thinking at all stages of problem solving.
- Allow for the fact that more than one strategy may be needed to solve a given problem and that problems may require original approaches.
- Present problem situations that closely resemble real situations in their richness and complexity so that the experience that students gain in the classroom will be transferable (p. 14).

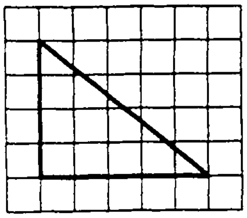
The 1991 *Mathematics Framework* (California State Department of Education, forthcoming), currently in draft form, builds on this foundation and moves significantly further in the directions suggested in this chapter. It recommends that lessons come in large, coherent chunks; that curricular units be anywhere from two to six weeks in length, be motivated by meaningful problems, and be integrated with regard to subject matter (e.g., containing problems calling for the simultaneous use of algebra and geometry, rather than having geometry taught as a separate subject, as if algebra did not exist); and that students engage in collaborative work, often on projects that take days and weeks to complete. Pilot projects for a radically new secondary curriculum, implementing these ideas for grades 9–11, began in selected California schools in September 1989.

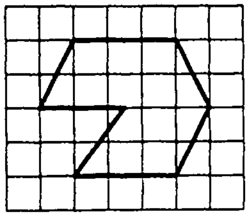
The presence of such projects, and their potential dissemination, raises significant practical and theoretical issues. For example, what kinds of teacher knowledge and behavior are necessary to implement such curricula on a large scale? One sees glimmers of ideas in the research (see Grouws & Cooney, 1989, for an overview), but in general, conceptions of how to teach for mathematical thinking have of necessity lagged behind our evolving concept of what it is to think mathematically. There are some signs of progress. For example, a small body of research (Peterson, Fennema, Carpenter, & Loef, 1989) suggests that with the appropriate in-service experiences (weeks of intensive study, not one-day workshops), teachers can learn enough about student learning to change classroom behavior. Much more research on teacher beliefs—how they are formed, how they can be made to evolve—is necessary. Also needed is research at the systemic level: What changes in school and district structures are likely to provide teachers with the support they need to make the desired changes in the classroom?

Offered as a conclusion is a brief discussion of what may be the single most potent systemic force in motivating change: assessment. *Everybody Counts* (National Research Council, 1989) states the case succinctly: "What is tested is what gets taught. Tests must measure what is most important" (p. 69). To state the case bluntly, current assessment measures (especially the standardized multiple-choice tests favored by many administrators for "accountability") deal with only a minuscule portion of the skills and perspectives encompassed by the phrase *mathematical power* and discussed in this chapter. The development of appropriate assessment measures, at both the individual and the school or district levels, will be a very challenging practical and theoretical task. Here are a few of the relevant questions:

What kinds of information can be gleaned from "open-ended questions," and what kinds of scoring procedures are reliable and informative both to those who do the assessing and those who are being tested? Here is one example of an interesting question type, taken from *A Question of Thinking* (California State Department of Education, 1989).

Imagine you are talking to a student in your class on the telephone and want the student to draw some figures. [They might be part of a homework assignment, for example]. The other student cannot see the figures. Write a set of directions so that the other student can draw the figures exactly as shown below.





To answer this question adequately, one must understand the geometric representation of the figures *and* be able to communicate using mathematical language. Such questions, while still rather constrained, clearly focus on goals other than simple subject matter "mastery." A large collection of such items would, at minimum, push the boundaries of what is typically assessed. But such approaches are only a first step. Two other approaches currently being explored (by the California Assessment Program, among others) are discussed next.

Suppose the student is asked to put together a portfolio representing his or her best work in mathematics. How can such portfolios be structured to give the best sense of what the student has learned? What kind of entries should be included ("the problem I am proudest of having solved," a record of a group collaborative project, a description of the student's role in a class project) and how can they be evaluated fairly?

Next, how can one determine the kinds of collaborative skills learned by students in a mathematics program? Suppose one picks four students at random from a mathematics class toward the end of the school year, gives them a difficult open-ended problem to work, and videotapes what the students do as they work on the problem for an hour. What kinds of inferences can one make, reliably, from the videotape? One claim is that a trained observer can determine within the first few minutes of watching the tape whether the students have had extensive experience in collaborative work in mathematics. Students who have not had such experience will most likely find it difficult to coordinate their efforts, while those who have often worked collaboratively will (one hopes) readily fall into certain kinds of cooperative behaviors. Are such claims justified? How can one develop reliable methods for testing them? Another claim is that students' fluency at generating a range of approaches to deal with difficult problems will provide information about the kinds of instruction they have received, and about their success at the strategic and executive aspects of mathematical behavior. But what kinds of information, and how reliable the information might be, is very much open to question.

In sum, the imminent implementation of curricula with ambitious pedagogical and philosophical goals will raise a host of unavoidable and fundamentally difficult theoretical and practical issues. It is clear that we have our work cut out for us, but it is also clear that progress over the past decade gives us at least a fighting chance for success.

References

Albers, D., & Alexanderson, J. (1985). *Mathematical people: Profiles and interviews*. Chicago: Contemporary books.

Alibert, D. (1988, June). Towards new customs in the classroom. *For the learning of mathematics, 8*(2), 31–35, 43.

Anderson, J. R. (1976). *Language, memory, and thought*. Hillsdale, NJ: Lawrence Erlbaum.

Anderson, J. R. (1983). *The architecture of cognition*. Cambridge: Harvard University Press.

Anderson, J. R. (1985). *Cognitive Psychology and its implications* (2nd ed.). New York: Freeman.

Balacheff, N. (1987). *Devolution d'un probleme et construction d'une conjecture: Le cas de "la somme des angles d'un triangle."* Cahier de didactique des mathematiques No. 39. Paris: IREM Universite Paris VII.

Ball, D. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Unpublished doctoral dissertation, Michigan State University.

Barwise, K. J., & Perry, J. (1983). *Situations and attitudes*. Cambridge, MA: MIT Press.

Bauersfeld, H. (1979). Hidden dimensions in the so-called reality of a mathematics classroom. In R. Lesh & W. Secada (Eds.), *Some theoretical issues in mathematics education: papers from a research presession* (pp. 13–32). Columbus, OH: ERIC.

Begle, E. (1979). *Critical variables in mathematics education*. Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Bloom, B. S. (1956). *Taxonomy of educational objectives. Handbook I: Cognitive domain. Handbook II: Affective domain*. New York: David McKay.

Brown, A. (1987). Metacognition, executive control, self-regulation, and other more mysterious mechanisms. In F. Reiner & R. Kluwe (Eds.), *Metacognition, motivation, and understanding* (pp. 65–116). Hillsdale, NJ: Lawrence Erlbaum.

Brown, J. S., & Burton, R. R. (1978). Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science, 2*, 155–192.

Brown, J. S., Collins, A., & Duguid, P. (1989, January-February). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32–42.

Burkhardt, H. (1988). Teaching problem solving. In H. Burkhardt, S. Groves, A. Schoenfeld, & K. Stacey (Eds.), *Problem solving—A world view (Proceedings of the problem solving theme group, ICME 5)* (pp. 17–42). Nottingham: Shell Centre.

Burkhardt, H., Groves, S., Schoenfeld, A., & Stacey, K. (Eds.). (1988). *Problem solving—A world view. (Proceedings of the problem solving theme group, ICME 5)*. Nottingham: Shell Centre.

California State Department of Education. (1985). *Mathematics framework for California public schools kindergarten through grade twelve*. Sacramento, CA: California State Department of Education.

California State Department of Education. (1989). *A question of thinking*. Sacramento, CA: State Department.

California State Department of Education. (forthcoming). *Mathematics framework for California public schools kindergarten through grade twelve.* Sacramento, CA: California State Department of Education.

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17–40). Hillsdale, NJ: Lawrence Erlbaum.

Carpenter, T. P., Lindquist, M. M., Matthews, W., & Silver, E. A. (1983). Results of the third NAEP mathematics assessment: Secondary School. *Mathematics Teacher. 76* (9), 652–659.

Carss, M. (Ed.). (1986). *Proceedings of the Fifth International Congress on Mathematics Education.* Boston: Birkhauser.

Charles, R., & Silver, E. A. (Eds.). (1988). *The teaching and assessing of mathematical problem solving.* Hillsdale, NJ: Lawrence Erlbaum.

Collins, A., Brown, J. S., & Newman, S. (1989). Cognitive apprenticeship: Teaching the craft of reading, writing, and mathematics. In L. B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser.* Hillsdale, NJ: Lawrence Erlbaum.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for research in mathematics education, 16*(5), 324–336.

Davis, P., & Hersh, R. (1981). *The mathematical experience.* Boston: Houghton-Mifflin.

deGroot, A. (1965). *Thought and choice in chess.* The Hague: Mouton

Descartes, R. (1952). *Rules for the direction of the mind* (E. S. Haldane and G. R. I. Ross, Trans.). In *Great Books of the Western World* (Vol. 31). Chicago: Encyclopedia Brittanica, Inc.

diSessa, A. (1983). Phenomenology and the evolution of intuition. In D. Gentner & A. Stevens (Eds.), *Mental Models* (pp. 15–33). Hillsdale, NJ: Lawrence Erlbaum.

Dossey, J., Mullis, I., Lindquist, M., & Chambers, D. (1988). *The mathematics report card: Are we measuring up? Trends and achievement based on the 1986 National Assessment.* Princeton, NJ: Educational Testing Service.

Duncker, K. (1945). *On problem solving.* Psychological Monographs 58, No. 5. (Whole # 270.) Washington, DC: American Psychological Association.

Ericsson, K., & Simon, H. (1980). Verbal reports as data. *Psychological review. 87*(3), 215–251.

Fawcett, H. P. (1938). *The nature of proof* (1938 Yearbook of the National Council of Teachers of Mathematics). New York: Columbia University Teachers College Bureau of Publications.

Flavell, J. (1976). Metacognitive aspects of problem solving. In L. Resnick (Ed.), *The nature of intelligence* (pp. 231–236). Hillsdale, NJ: Lawrence Erlbaum.

Flavell, J. H., Friedrichs, A. G., & Hoyt, J. D. (1970). Developmental changes in memorizations processes. *Cognitive psychology 1*, 323–340.

Garofalo, J., & Lester, F. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for research in mathematics education, 16*(3), 163–176.

Geertz, C. (1983). *Local knowledge.* New York: Basic Books.

Greeno, J. (1988). For the study of mathematics epistemology. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 23–31). Reston, VA: National Council of Teachers of Mathematics.

Groner, R., Groner, M., & Bischof, W. (Eds.). (1983). *Methods of heuristics.* Hillsdale, NJ: Lawrence Erlbaum.

Grouws, D., & Cooney, T. (Eds.) (1988). *Effective mathematics teaching.* Hillsdale, NJ: Lawrence Erlbaum.

Groves, S., & Stacey, K. (1984). *The Burwood Box.* Melbourne, Australia: Victoria College, Burwood.

Hadamard, J. (1945). *An essay on the psychology of invention in the mathematical field.* Princeton: Princeton University Press.

Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly, 87*, 519–524.

Harvey, J. G., & Romberg, T. A. (1980). *Problem-Solving studies in mathematics.* Madison, WI: Wisconsin Research and Development Center Monograph Series.

Hayes-Roth, B., & Hayes-Roth, F. (1979). A cognitive model of planning. *Cognitive Science, 3*, 275–31.

Heller, J., & Hungate, H. (1985). Implications for mathematics instruction of research on scientific problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 83–112). Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J. (1985). (Ed.) *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Hinsley, D.A., Hayes, J. R., & Simon, H. A. From words to equations: meaning and representation in algebra word problems. In M. Just & P. Carpenter (Eds.), *Cognitive processes in comprehension* (pp. 89–106). Hillsdale, NJ: Lawrence Erlbaum.

Hoffman, K. (1989, March). *The science of patterns: A practical philosophy of mathematics education.* Paper presented to the Special Interest Group for Research in Mathematics Education at the 1989 Annual Meeting of the American Educational Research Association, San Francisco.

Hunt, E. (1975). *Artificial intelligence.* New York: Academic Press.

International Association for the Evaluation of Educational Achievement. (1987). *The underachieving curriculum: assessing U.S. school mathematics from an international perspective.* Champaign, IL: Stipes Publishing Company.

James, W. (1890). *Principles of Psychology (2 volumes).* New York: Holt.

Janvier, C. (Ed.). (1897). *Problems of representation in the teaching and learning of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for research in mathematics education, 8*, 163–180.

Kaput, J. (1985). Representation and problem solving: Issues related to modeling. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 381–398). Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. (1989). Linking representations in the symbol system of algebra. In S. Wagner & C. Kieran (Eds.), *Research Issues in the learning and teaching of algebra* (pp. 167–194). Hillsdale, NJ: Lawrence Erlbaum.

Karmiloff-Smith, A. (1979) Problem solving construction and representations of closed railway circuits. *Archives of psychology, 47*, 37–59.

Kilpatrick, J. (1967). Analyzing the solution of word problems in mathematics: An exploratory study. (Unpublished doctoral dissertation, Stanford University). *Dissertation Abstracts International, 28*, 4380A. (University Microfilms 68-5, 442.)

Kilpatrick, J. (1978). Variables and methodologies in research on problem solving. In L. Hatfield (Ed.), *Mathematical problem solving* (pp. 7–20). Columbus, OH: ERIC.

Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching mathematical problem solving. In E. A. Silver, *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–16). Hillsdale, NJ: Lawrence Erlbaum.

Kitcher, P. (1984). *The nature of mathematical knowledge.* New York: Oxford University Press.

Krulik, S. (Ed.). (1980). *Problem solving in school mathematics.* (1980 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: NCTM.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children* (J. Teller, Trans; J. Kilpatrick, & I. Wirszup, Eds.). Chicago: University of Chicago Press.

Lakatos, I. (1977). *Proofs and refutations* (revised ed.). Cambridge: Cambridge University Press.

Lakatos, I. (1978). *Mathematics, science, and epistemology.* Cambridge: Cambridge University Press.

Lampert, M. (1990). When the problem is not the problem and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27* (1), 29-63.

Lave, J. (1988). *Cognition in practice.* Boston: Cambridge University Press.

Lave, J. (in preparation). *Tailored learning: Apprenticeship and everyday practice among craftsmen in West Africa.*

Lave, J., Smith, S., & Butler, M. (1988). Problem solving as everyday practice. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 61–81). Reston, VA: National Council of Teachers of Mathematics.

Lave, J., & Wenger, E. (1989). *Situated learning: Legitimate peripheral participation.* (Manuscript available from author, School of Education University of California, Berkeley.)

Lesh, R. (1983). *Metacognition in mathematical problem solving.* Unpublished manuscript. Available from author, Educational Testing Service, Rosedale Road Princeton, NJ.

Lesh, R. (1985). Conceptual analyses of problem solving performance. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 309–330). Hillsdale, NJ: Lawrence Erlbaum.

Lester, F., Garofalo, J., & Kroll, D. (1989). *The role of metacognition in mathematical problem solving: A study of two grade seven classes.* Final report to the National Science Foundation of NSF project MDR 85-50346.

Lucas, J. (1974). The teaching of heuristic problem-solving strategies in elementary calculus. *Journal for reseach in mathematics education, 5,* 36–46.

Mason, J., Burton, L., & Stacey, K. (1982) *Thinking mathematically.* New York: Addison Wesley.

Mayer, R. (1985). Implications of cognitive psychology for instruction in mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Mulitiple research perspectives* (pp. 123–138). Hillsdale, NJ: Lawrence Erlbaum.

McLeod, D., & Adams, V. (1989). *Affect and mathematical problem solving: A new perspective.* New York: Springer-Verlag.

Mead, G. H. (1934). *Mind, self, and society.* Chicago: University of Chicago Press.

Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review 63,* 81–97.

Milne, W. J. (1897). *A mental arithmetic.* New York: American Book.

Minsky, M. (1961). Steps toward artificial intelligence. *Proceedings of the institute of radio engineers, 49,* 8–30.

Minsky, M. (1975). A framework for representing knowledge. (1977). In P. Winston (Ed.), *The psychology of computer vision* (pp. 170–195). New York: McGraw-Hill.

Moschkovich, J. (1989). *Constructing a problem space through appropriation: A case study of tutoring during computer exploration.* Paper presented at the 1989 annual meetings of the American Educational Research Association, San Francisco.

National Assessment of Educational Progress. (1983). *The third national mathematics assessment: Results, trends, and issues* (Report No. 13-MA-01). Denver, CO: Educational Commission of the States.

National Center of Educational Statistics. (1988a). *Trends in minority enrollment in higher education, Fall 1986-Fall 1988.* Washington, DC: U.S. Department of Education.

National Center of Educational Statistics. (1988b). *Digest of education statistics, 1988.* Washington, DC: U.S. Department of Education.

National Commission on Excellence in Education. (1983). *A nation at risk: The imperative for educational reform.* Washington, DC: U.S. Government Printing Office.

National Council of Supervisors of Mathematics. (1977, October). Position paper on basic mathematical skills. *Arithmetic teacher 25,* 19–22.

National Council of Teachers of Mathematics. (1980). *An agenda for action.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

National Research Council. (1990a). *Reshaping school mathematics: A philosophy and framework for curriculum.* Washington, DC: National Academy Press.

National Research Council. (1990b). *A challenge of numbers.* Washington, DC: National Academy Press.

Newell, A. (1983) The heuristic of George Pólya and its relation to artificial intelligence. In R. Groner, M. Groner, & W. Bischof (Eds.), *Methods of heuristics* (pp. 195–243). Hillsdale, NJ: Lawrence Erlbaum.

Newell, A., & Simon, H. (1972). *Human problem solving.* Englewood cliffs, NJ: Prentice-Hall.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school.* Cambridge: Cambridge University Press.

Norman, D. (Ed.). (1970). *Models of human memory.* New York: Academic Press.

Novak, J. (Ed.). (1987). *Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics.* Ithaca, NY: Cornell University.

Oxford University Press. *Oxford English Dictionary* (compact ed.). Oxford: Author.

Palincsar, A., & Brown, A. (1984). Reciprocal teaching of comprehension-fostering and comprehension-monitoring activities. *Cognition and Instruction, 1* (2), pp. 117–175.

Pavlov, I. P. (1928). *Lectures on conditioned reflexes* (3rd ed.). (W. H. Gantt, Trans.) New York: International Publishers.

Pea, R. (1989). *Socializing the knowledge transfer problem.* (IRL report IRL 89-0009). Palo Alto, CA: Institute for Research on Learning.

Peters, R. S. (1962) *Brett's history of psychology* (edited and abridged by R. S. Peters). Cambridge, MA: The M.I.T. Press.

Peterson, P., Fennema, E., Carpenter, T., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and instruction 6*(1), 1–40.

Piaget, J. (1928). *The language and thought of the child.* New York: Harcourt Brace.

Piaget, J. (1930). *The child's conception of physical causality.* New York: Harcourt Brace.

Piaget, J. (1954). *The construction of reality in the child* (M. Cook, trans.) New York: Ballantine Books.

Piaget, J. (1971). *The child's conception of time.* (original French version published 1927). New York: Ballantine Books.

Poincaré, H. (1913). *The foundations of science* (G. H. Halstead, Trans.). New York: Science Press.

Pollak, H. (1987). Cognitive science and mathematics education: A mathematician's perspective. In Schoenfeld, A. H. (Ed.), *Cognitive science and mathematics education* (pp. 253–264). Hillsdale, NJ: Lawrence Erlbaum.

Pólya, G. (1945; 2nd edition, 1957). *How to solve it.* Princeton: Princeton University Press.

Pólya, G. (1954). *Mathematics and plausible reasoning; Vol. 1. Induction and analogy in mathematics; Vol. 2. Patterns of plausible inference.* Princeton: Princeton University Press.

Pólya, G. (1962,1965/1981). *Mathematical Discovery* (Volume 1, 1962; Volume 2, 1965). Princeton: Princeton University Press. (Combined paperback edition, 1981). New York: Wiley.

Pólya, G., & Szego G. (1925) *Aufgaben und Lehrsätze aus der Analysis I*. Berlin, Germany: Springer. An English version, *Problems and theorems in analysis I* (D. Aeppli, Trans.), was published by Springer (New York) in 1972.

Putnam, R. T., Lampert, M., & Peterson, P. (1989). Alternative perspectives on knowing mathematics in elementary schools. Elementary subjects center series number 11. Michigan State University: Center for learning and teaching elementary subjects.

Resnick, L. (1988). Treating mathematics as an ill-structured discipline. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 32–60). Reston, VA: National Council of Teachers of Mathematics.

Rissland, E. (1985). Artificial intelligence and the learning of mathematics: A tutorial sampling. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 147–176). Hillsdale, NJ: Lawrence Erlbaum.

Rogoff, B., & Lave, J. (Eds.). (1984). *Everyday cognition: Its development in social context.* Cambridge: Harvard University Press.

Romberg, T., & Carpenter, T. (1986). Research in teaching and learning mathematics: Two disciplines of scientific inquiry. In M. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed. pp. 850–873). New York: Macmillan.

Ryle, G. (1949). *The concept of mind.* London: Hutchinson.

Sacerdoti, E. (1974) Planning in a hierarchy of abstraction spaces. *Artificial intelligence 5,* 115–136.

Scardamalia, M., & Bereiter, C. (1983). Child as co-investigator: Helping children to gain insight into their own mental processes. In S.G. Paris, M. Olson, & H.W. Stevenson (Eds.), *Learning and Motivation in the Classroom* (pp. 61–82). Hillsdale, NJ: Lawrence Erlbaum.

Schank, R., & Abelson, R. (1977). *Scripts, plans, goals, and understanding.* Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. (1983). *Problem solving in the mathematics curriculum: A report, recommendations, and an annotated bibliography.* Washington, D.C.: Mathematical Association of America.

Schoenfeld, A. (1985a). *Mathematical problem solving.* New York: Academic Press.

Schoenfeld, A. (1985b). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361–380). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. (1987a). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 189–215). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. (1987b, December). Pólya, problem solving, and education. *Mathematics magazine, 60*(5), 283–291.

Schoenfeld, A. (1988, Spring). When good teaching leads to bad results: the disasters of "well taught" mathematics classes. *Educational psychologist, 23*(2), 145–166.

Schoenfeld, A. (1989a). Problem solving in context(s). In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving,* (pp. 82–92). Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. (1989b). Explorations of students' mathematical beliefs and behavior. *Journal for research in mathematics education. 20*(4), 338–355.

Schoenfeld, A. (1989c). Ideas in the air: Speculations on small group learning, environmental and cultural influences on cognition, and epistemology. *International Journal of Educational Research, 13*(1), 71–88.

Schoenfeld, A. (1989d). Teaching mathematical thinking and problem solving. In L. B. Resnick & B. L. Klopfer (Eds.), *Toward the thinking curriculum: Current cognitive research* (pp. 83–103). (1989 Yearbook of the American Society for Curriculum Development). Washington, DC: ASCD.

Schoenfeld, A. (Ed.) (1990a) *A sourcre book for college mathematics teaching.* Washington, DC: Mathematical Association of America.

Schoenfeld, A. (1990b). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In D.N. Perkins, J. Segal, & J. Voss (Eds.), *Informal reasoning and education* (pp. 281-300). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. (in preparation). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving.*

Schoenfeld, A., Smith, J., & Arcavi, A. (in press). Learning: The microgenetic analysis of one student's understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4). Hillsdale, NJ: Lawrence Erlbaum.

Shaughnessy, M. (1985). Problem-solving derailers: The influence of misconceptions on problem solving performance. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 399–416). Hillsdale, NJ: Lawrence Erlbaum.

Shell Centre for Mathematical Education. (1984). *Problems with patterns and numbers.* Nottingham, England: Shell Centre.

Silver, E. A. (1979). Student perceptions of relatedness among mathematical verbal problems. *Journal for research in mathematics education, 10*(3), 195–210.

Silver, E. A. (1981). Recall of mathematical problem information: Solving related problems. *Journal for research in mathematics education, 12*(1), 54–64.

Silver, E. A. (1982). *Thinking about problem solving: Toward an understanding of metacognitive aspects of problem solving.* Paper presented at the Conference on Thinking, Suva, Fiji, January.

Silver, E. A. (Ed.). (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem solving instruction. In A. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 33–60). Hillsdale, NJ: Lawrence Erlbaum.

Silver, E. A., Branca, N., & Adams, V. (1980). Metacognition: The missing link in problem solving? In R. Karplus (Ed.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 429–433). Boston: Birkhäuser.

Simon, H. (1979). Information processing models of cognition. *Annual Review of Psychology, 30,* 363–96.

Simon, H. (1980). Problem solving and education. In D. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research* (pp. 81–96). Hillsdale, NJ: Lawrence Erlbaum.

Skinner, B. F. (1974). *About behaviorism.* New York: Knopf.

Smith, J. P. (1973). The effect of general versus specific heuristics in mathematical problem solving tasks. *Dissertation Abstracts International, 34,* 2400A. (University Microfilms 73-26, 637.)

Sowder, L. (1985). Cognitive psychology and mathematical problem solving: A discussion of Mayer's paper. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Mulitiple research perspectives* (pp. 139–145). Hillsdale, NJ: Lawrence Erlbaum.

Stanic, G., & Kilpatrick, J. (1988). Historical perspectives on problem solving in the mathematics curriculum. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1–22). Reston, VA: National Council of Teachers of Mathematics.

Steen, L. (1988, April 29). The science of patterns. *Science, 240*, 611–616.

Stevenson, H. W., Lee, S-Y., & Stigler, J. W. (14 February 1986). Mathematics Achievement of Chinese, Japanese, and American Children. *Science, 231*, 693–698.

Stigler, J., & Perry, M. (1989). Cross cultural studies of mathematics teaching and learning: Recent findings and new directions. In D. Grouws & T. Cooney (Eds.), *Effective Mathematics Teaching* (pp. 194–223). Hillsdale, NJ: Lawrence Erlbaum.

Stipek, D. J., & Weisz, J. R. (1981). Perceived personal control and academic achievement. *Review of Educational Research 51*(1), 101–137.

Stodolsky, S. S. (1985). Telling math: Origins of math aversion and anxiety. *Educational Psychologist 20*, 125–133.

Suinn, R. M., Edie, C. A., Nicoletti, J., & Spinelli, P. R. (1972). The MARS, a measure of mathematics anxiety: Psychometric data. *Journal of clinical psychology, 28*, 373–375.

Thompson, A. (1985). Teachers' conceptions of mathematics and the teaching of problem solving. In E. A. Silver, *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 281–294). Hillsdale, NJ: Lawrence Erlbaum.

Thorndike, E. L. (1924). Mental discipline in high school studies. *Journal of Educational Psychology, 15*, 1–22, 83–98.

Thorndike, E. L., & Woodworth, R. S. (1901). The influence of improvement in one mental function on the efficiency of other mental functions (I). *Psychological Review, 8*, 247–261.

Tobias, S. (1978). *Overcoming math anxiety*. New York: W. W. Norton.

Vygotsky, L. (1978). *Mind in Society*. Cambridge: Cambridge University Press.

Wagner, S., & Kieran, C. (Eds.). (1989). *Research Issues in the learning and teaching of algebra*. Hillsdale, NJ: Lawrence Erlbaum.

Wallas, G. (1926). *The art of thought*. Selections in P.E. Vernon (Ed., 1970), *Creativity*, Middlesex, England: Penguin, p. 91–97.

Watson, J. (1930). *Behaviorism* (2nd ed.). New York: Norton.

Watson, R. I. (1978). *The great psychologists*. Philadelphia: Lippincott.

Webster's. (1979) *New Universal Unabridged Dictionary*. Second edition. New York: Simon & Schuster.

Wertheimer, M. (1945/1959). *Productive thinking*. New York: Harper and Row.

Wilson, J. (1967) Generality of heuristics as an instructional variable. *Dissertation Abstracts International, 28*. 2575A. (University Microfilms 67-17, 526).

Wundt, W. (1904). *Principles of physiological psychology* (5th German ed., Vol. 1). (E. B. Titchener, Trans.). New York: Macmillan.

Zweng, M., Green, T., Kilpatrick, J., Pollak, H., & Suydam, M. (Eds.). (1983). *Proceedings of the Fourth International Congress on Mathematics Education*. Boston: Birkhauser.