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20

Exploring Generalization through Pictorial Growth Patterns

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WHEN asking prospective and practicing elementary school teachers to describe algebraic thinking at the beginning of a university course or professional development workshop, both give similar, yet limited descriptions. They equate algebraic thinking with algebra, viewing it as a subject that requires solving equations, using variables, and learning or memorizing rules for manipulating expressions. This viewpoint is a "traditional image of algebra" (Kaput 1999). As Smith (2003) observes, the term *algebra* is typically equated with a study of symbol systems, whereas *algebraic thinking* is a broader term, used "to indicate the kinds of generalizing that precede or accompany the use of algebra" (p. 138).

The study of algebra as conceptualized by *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000) focuses on algebraic thinking and "emphasizes relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change" (p. 37). In thinking about algebra in the elementary grades, it is essential to focus on aspects of algebraic thinking, reflecting on how generalizing (Kaput 1999) and analyzing change (Smith 2003) lie at the heart of algebraic thinking.

Fostering algebraic thinking in elementary school classrooms establishes a firm foundation for students to draw on as they begin a formalized study of algebra in the later grades. Since elementary school teachers often believe they do not have time to explore algebraic thinking in their classrooms, my goal in professional development is to help teachers see how algebraic thinking can be infused into the curriculum without becoming a separate topic. Teachers require a personal understanding of what it means to think algebraically before they can promote algebraic understanding in their classrooms. They need multiple experiences analyzing change and identifying, representing, and generalizing relationships among variables.

In this article, four tasks will be described that involve pictorial growth patterns used with both prospective and practicing teachers to challenge them to think algebraically. Three tools to help teachers grow in their abilities to generalize relationships will be examined, and suggestions for promoting flexibility in the use of these tools will be given.

What Is a Pictorial Growth Pattern?

The swimming pool problem (Ferrini-Mundy, Lappan, and Phillips 1997) and the V-pattern, from the Mathematics in Context curriculum (National Center for Research in Mathematical Sciences and Freudenthal Institute 1998, p. 6), shown in figure 20.1, are examples of pictorial growth patterns, also called geometric patterns (NCTM 2000). A pictorial growth pattern is a sequence of figures in which the objects in the figure change from one term to the next, usually in a predictable way. A pictorial growth pattern typically involves two variables; some quantifiable aspect of this pictorial pattern of objects (the dependent variable) is coordinated with an indexing or counting system (the independent variable) that provides an identification of the position of the figure in the pattern. For example, in figure 20.1, the independent variable for both patterns is a counting sequence. Often, the dependent variable is the total number of some aspect of the figure, such as the total number of dots in the V-pattern or the total number of tiles in the Swimming Pool pattern. Other aspects of the pictorial growth pattern can be chosen for the dependent variable, such as the *number of border tiles* or the *number of pool tiles* in the Swimming Pool pattern.

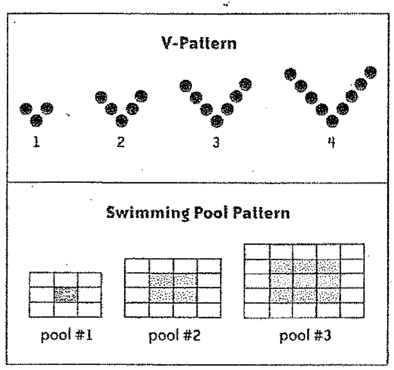


Fig. 20.1. Examples of pictorial growth patterns

Tools for Expressing Generalizations

Pictorial growth patterns are a rich context for exploring generalization, a major component of algebraic thinking (Kaput 1999). Teachers analyze, describe, and extend pictorial growth patterns with the ultimate goal of *generalizing* relationships in these patterns. Teachers can use many different tools to form generalizations, including the physical constructions of a pattern, algebraic symbols, and an explicit analysis of change.

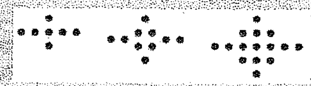
Tool 1: The Physical Construction of a Pattern

A typical school approach for analyzing pictorial growth patterns is to translate a quantifiable aspect of the pictorial pattern into a number pattern and use the numerical data as a basis for analysis, thus adding an extra step of conversion into the process. By immediately translating the diagram to a numeric representation, one loses the opportunity to relate the numerical relationships directly to the context and to the physical construction of the pattern, and many crucial insights are lost (Orton, Orton, and Roper 1999). It stands to reason, then, that "a much more constructive approach [than immediate number conversion] is to ask students to build one element of the pattern physically and explain how it is put together, not in terms of number but in terms of its underlying physical structure" (Thornton 2001, p. 389). Students who analyze the physical structure or construction of a pictorial growth pattern often interpret the generalized relationship inherent in it. This focus on relationships among varying quantities can lead to a correct symbolic representation of the generalization (Orton, Orton, and Roper 1999; Thornton 2001).

The representation of a pictorial growth pattern is very useful in and of itself in promoting the analysis and generalization of relationships. I have introduced teachers to pictorial growth patterns using the Diamond-Dot task (Billings and Wells 2005) in figure 20.2, initially created for use in a teacher workshop exploring algebra concepts. This activity helps teachers to establish what constitutes a pictorial pattern and to intentionally use the physical construction of the pattern to analyze, extend, and generalize it. I discourage teachers from using numeric tables to analyze relationships between variables directly in pictorial growth patterns. After they can comfortably generalize a relationship directly from the physical construction of the pattern in a variety of contexts, we then explore and analyze the numeric representation of the pattern.

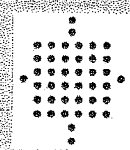
The first four questions of this activity encourage the teachers to directly examine the alignment of the dots in the pattern. By physically drawing the fourth diamond and by determining whether the figure in Question 2 fits into the pattern further in the sequence, the teachers need to analyze the diamond's dot construction. For example, some teachers claimed that the given diamond fit the pattern because "it has the correct number of dots for the sixth diamond" or "it fits because it is the sixth diamond rotated clockwise 90 degrees." Others suggested that it did not fit the pattern because "the extra dots aren't properly placed around the center square" but agreed that if they rotated the diamond so that the two extra dots stick out from the right and left of the square (rather than top and bottom), it would represent the

Below is a sequence of diamonds made from dots:



Diamond #1 Diamond #2 Diamond #3

1. Draw the fourth diamond in the sequence.
2. Would the following diamond fit into the given sequence? If yes, how can you tell and which diamond number would it be? If no, why not?



3. Explain how you would draw the tenth diamond. How many dots would it have?
4. How many dots would the 25th diamond have? Explain how you know.
5. Describe with words a rule that would enable you to determine the number of dots needed to make any diamond in the sequence.

Fig. 20.2. Diamond-dot pattern

sixth diamond. Although they initially disagreed, this type of discussion helped the teachers clarify what constitutes a pattern; the number as well as the direction and placement of the dots is crucial for continuing the pattern. By explaining *how* to draw the *n*th diamond in Question 3, teachers must extend the pattern and once again examine the physical construction of the figure. Thus, it is natural for teachers to continue using the physical construction of the pattern to predict that there are 631 dots in the 25th diamond (Question 4). One teacher explained:

> You make a 25 by 25 square of dots, which gives you 625 dots, and then add 6 to that to get 631 dots. The 6 comes from the 4 dots sticking out on the right and left of the square, two on each side, plus two more for the one dot on the top and one dot on the bottom of the square.

Through this extending process, the teachers shifted their focus to analyzing the relationship between the diamond number and the number of dots needed to make this diamond. Because they spent time analyzing the physical construction of the figures, they could identify the parts of the figure that stay the same from one diamond to the next (the six dots placed around the center square) and the parts of the diamond that change (the size of the center square made from an array of dots).

They then used this information to state a generalization for determining the number of dots in any diamond in the sequence.

Tool 2: Algebraic Symbols

Competence in algebra assumes facility with certain symbolic notation systems, including the use of symbols to record ideas and provide insights into mathematical situations. The use of algebraic symbols to represent and analyze mathematical situations is one of the four primary components of the Algebra Standard (NCTM 2000). Variables, expressions, and equations become shorthand tools for describing the relationships that emerge as teachers analyze, extend, and generalize pictorial growth patterns (Smith 2003).

Teachers should first verbalize generalizations, since representing a relationship with words is often easier than finding a symbolic generalization for this same relationship (English and Warren 1998). Articulating the generalization verbally builds a foundation for constructing a symbolic representation.

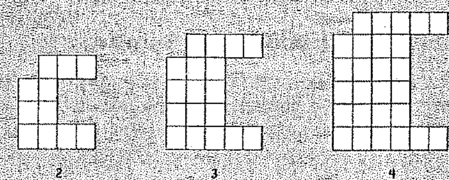
The pile growth pattern activity (fig. 20.3), adapted from materials developed through the Michigan Mathematics Middle School Reform project (Tucher et al. 2003) provides a context for using algebraic symbols as tools. In this activity, teachers are given a pattern that does not start with the first figure, and they need to determine what initial figures in this pattern look like. Next, they are asked to extend and analyze the pattern and then determine a general rule for describing the growth pattern, expressing this rule verbally and symbolically. As teachers find additional rules to describe the relationship and compare these rules, they build an informal understanding of the simplification and equivalence of expressions.

Asking "Backwards" Questions

Asking "backwards" questions provides opportunities for teachers to think more flexibly about ways to create, interpret, and represent generalizations. Backwards questions are questions that reverse the focus of the task; they may require the problem solver to begin analyzing a situation from a different starting point than the problem solver is used to or to make connections among representations or tools in different orders. Asking teachers backwards questions encourages them to continue to think about the patterns and generalizations inherent in these patterns from multiple perspectives. For example, the first question of the pile growth pattern is a backwards question. As well as analyzing how the piles continue to grow larger, teachers must also extend the pattern of piles in a backward direction to create smaller piles earlier in the sequence. This backwards question continues to challenge the teachers' understanding and facility to analyze a pictorial growth pattern using the physical construction of the figure.

To answer this backwards question, many teachers created the first pile (see fig. 20.4) by taking away tiles from pile 2 in the opposite manner in which they added

Examine the following pattern of "piles."



1. Sketch and label the fifth, sixth, first, and 0th pile on grid paper.
2. How many square tiles are needed to construct each of these piles?
3. Describe with a written explanation how you could *sketch or construct* the 100th pile.
4. *Using the model or picture directly*, describe *with words* at least three *different* ways you could determine the number of tiles needed to make the *p*th pile in the sequence.
5. If you did not already do so, write a rule or formula that matches *each* of the ways you described in #4. Each rule (symbolic representation) should allow you to determine easily the number of tiles needed to make the *p*th pile in the sequence. Define your variables.

Fig. 20.3. Pile growth pattern

tiles to pile 4 to build pile 5. However, they struggled to continue to reverse this mental process when removing tiles from pile 1 to create pile 0. Typically, teachers constructed pile 0 in one of the two ways found in figure 20.4. In both instances, the teachers used the physical construction of the pattern as a tool to create the 0th pile.

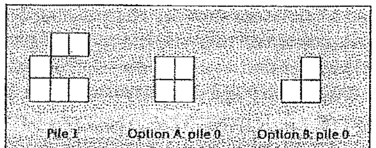


Fig. 20.4. Teachers' constructions of pile 1 and pile 0

Teachers who proposed option A often focused on the four tiles to the far right of each figure, identifying them as remaining constant. One teacher articulated, "There are always four extra tiles, so I kept the two on top and the two on bottom. To follow the pattern, there would be zero columns of tiles to the left of that." Teachers who proposed option B typically arrived at this visual representation by reasoning directly about the pattern instead of reversing their thinking; they extended the pattern to larger values for p (the pile number), found a generalization of the relationship based on the physical construction of the figures, and used this generalization to create the 0th pile. Since this was the strategy used by the majority of the teachers, we postponed our discussion of what the 0th pile looked like until after all teachers had ample opportunity to create their own generalizations. Teachers have occasionally suggested option B by reversing the construction process. For example, when analyzing consecutive terms in a backward direction, one teacher noted that the number of tiles in the leftmost column in the pile decreases by 1. She reasoned that since there are two tiles in the leftmost column of pile 1, there should be one tile in the leftmost column of the 0th pile. This teacher then observed that the number of columns in the piles also decreases by 1 between consecutive terms and reasoned that the 0th pile must have two columns of tiles. Likewise, the number of rows decreases by one, so the 0th pile must have two rows. Combining these observations, the teacher concluded that the pile would look like option B.

Using Algebraic Symbols as Tools to Represent a Generalization

As the teachers continued analyzing this pictorial growth pattern to answer questions 4 and 5, they partitioned the construction of a pile in different ways, which led to a variety of generalizations and rules. Figure 20.5 includes computer-generated versions illustrating the thinking behind six of the most common rules and sketches that emerged as teachers shared their rules in a large group discussion. Once again, the teachers needed time to explore *how* the pattern was physically constructed, observing how piles change and stay the same. As they extended the pattern and identified rules for predicting the total number of tiles needed for any pile, they naturally used symbols as a shorthand method for recording their generalizations.

As teachers shared their generalizations, we discussed how the generated symbolic rules recorded the *process* used to create the generalization. In order to establish an intentional record of how a generalization was derived, we did not simplify the rules. Often the rules were written in nonstandard forms. As each new rule and sketch was recorded on a whiteboard, we compared the process of partitioning the physical construction of the pile with the rule. For example, although all the expressions in figure 20.5 simplify to $p^2 + 2p + 3$, only one rule (b) illustrates how the pile may be visualized as a p by p square, with two rows of length p tiles and 3 "extra" tiles. As we discussed the many rules that emerged, teachers became more comfortable using variables and expressions to represent their generalizations. They could view the symbolic representation as a tool for recording a rule and the process used to construct it.

After establishing these different symbolic generalizations, teachers were encouraged to describe how they used their generalizations to create representations

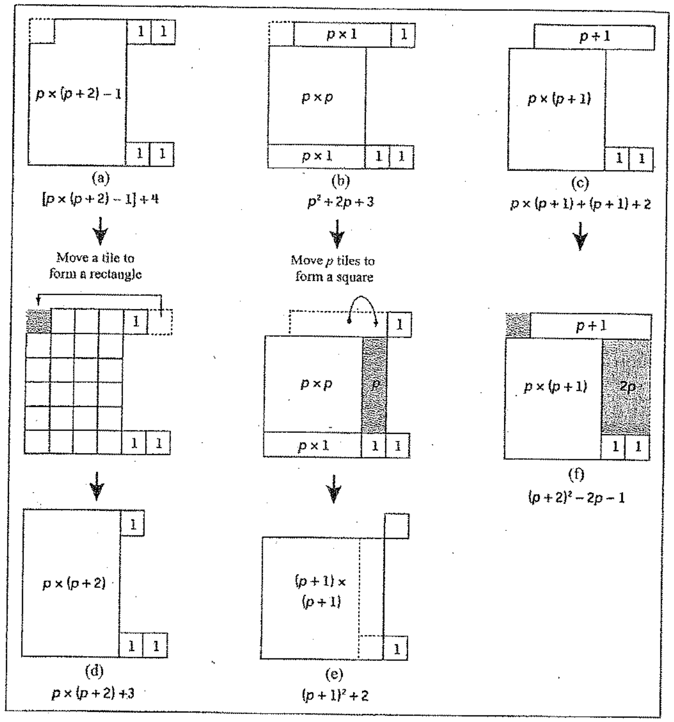


Fig. 20.5. Predicting the number of tiles in the *p*th pile

of the 0th pile. For example, one teacher explained why she thought the 0th pile looked like option B (see fig 20.4). "There is a 0 by 2 rectangle in the center but it has 0 tiles. I have three leftover tiles and put one on top, shifted over so it looks like the other piles, and the two below it." She viewed the construction of the pile as $3 + p \times (p + 2)$ (see fig. 20.5[d]). Another used the $p^2 + 2p + 3$ construction of the pile in figure 20.5(b) to justify why the 0th pile matches option B. He stated, "We have a 0 by 0 center square, a row of 0 + 1 tiles on top of this square, and a row of 0 + 2 tiles below it. Since there is no square, these two rows are squished together and we put the top row so that the upper left corner tile is missing." As we ended our discussion, the teachers ultimately concluded that option B (in fig. 20.4) was the most appropriate representation of the 0th pile because it matched the ways they had extended and generalized the relationship in this pattern.

Tool 3: An Explicit Analysis of Change

Analyzing change is a crucial aspect of thinking algebraically (NCTM 2000; Smith 2003) and is inherent to the process of analyzing, extending, and generalizing relationships found in pictorial growth patterns. However, teachers need an explicit awareness of how to identify and analyze change, and by isolating the analysis of change as a distinct tool, teachers are encouraged to intentionally reflect on change. They can then explicitly apply this knowledge as a tool for generalizing relationships.

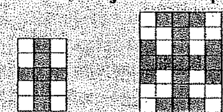
Tasks like those found in figures 20.2 and 20.3 provide opportunities for teachers to analyze the change in a pattern by examining the physical construction of the figures; they analyze how the figures in the pattern physically change using two perspectives—how consecutive figures in a pictorial pattern change (a covariation analysis of change) and how aspects of the figure stay the same and change as the pattern continues. As teachers coordinate the figure number with the changing aspect of figure (a correspondence analysis of change), using the physical construction of the figures in the pattern to help guide this analysis, they are usually successful in generalizing the underlying relationship. In order to continue to extend teachers' abilities to analyze change, it is beneficial to ask teachers to engage in patterning tasks, such as the one found in figure 20.6, that encourage the creation of pictorial growth patterns. In Part 1 of this activity, only two figures of the pattern are given. As a result, there are numerous ways to extend the pattern. Item 1 encourages teachers to analyze change explicitly from a different perspective; they need to create their own pictorial growth pattern, using manipulative square tiles or grid paper, by systematically extending each term in the sequence.

Many teachers struggled as they extended the tile pattern. For example, some immediately began to construct tile #3 to "look like it will fit the pattern" *before* analyzing how the square tiles change between consecutive tile designs. However, since the teachers also needed to construct tiles #4 and #6, they typically reanalyzed their approaches, realizing the need to examine *how* and *if* the consecutive tile designs they created physically changed in predictable ways. Also, they began analyzing what stays the same and what changes between consecutive tile designs in their patterns. As teachers analyzed and constructed covariational and correspondence types of change, they could then construct a pictorial growth pattern.

Asking "Backwards" Questions to Develop Flexibility to Use Tools

Asking backwards questions, an idea described earlier in this article, challenges teachers to continue to grow in their abilities to use tools (physical constructions, algebraic symbols, and explicit analysis of change) flexibly to interpret and create generalizations. For example, Part I of the Make Your Own Tile activity (fig. 20. 6),

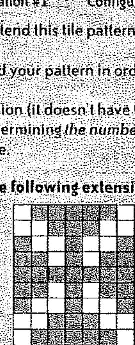
Part I: Here are the first two tile configurations in a pictorial growth pattern.



Configuration #1 Configuration #2

1. Show *two* different ways to extend this tile pattern. Draw the third and fourth tile designs on grid paper.
2. Explain how you would extend your pattern in order to create or draw the sixth tile design in the pattern.
3. Choose a "favorite" tile extension (it doesn't have to be yours). *Using the model or picture*, describe a rule for determining *the number of square tiles* needed to make any tile design in the sequence.

Part II: Consider the following extension of the tile pattern.



Configuration #3

1. *Using the physical construction of the tile pattern*, describe at least *two different* rules for determining *the number of tiles* needed to make any figure in the sequence.
2. Esther, Pam, and Steve each found a different rule for generalizing the number of square tiles needed to build the *n*th configuration. Explain how each student partitioned the configuration in order to come up with this rule. Feel free to use a diagram to help explain their rules.
   Esther's Rule: $(2n - 1)(2n + 1) + 4$ = # of tiles needed for *n*th figure where n = figure #.
   Pam's Rule: $3 + 4(n/2)(2n)$ = # of tiles needed for nth figure where n = figure #.
   Steve's Rule: $[(2n + 1)(2n + 3)] - (2n \times 4)$ = # of tiles needed for nth figure where n = figure #.

Fig. 20.6. Make your own tile pattern.

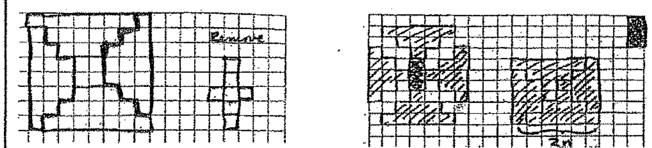
provides an opportunity for teachers to analyze change from a backwards perspective; instead of analyzing a given change, they *construct* change to extend a pattern systematically.

In Part II of this same tile activity, item 2 asks teachers to analyze the situation from a backwards perspective; start with the a symbolic rule and use this representation to design the physical construction that matches the rule. However, before the teachers are asked to make sense of given rules and connect these rules to the physical constructions of the patterns, they first need to create their own generalizations.

Teachers typically find generalizing the relationship in Part II, item 1 particularly challenging, since the figures grow in an "increasingly increasing" way. As teachers analyze change and use the physical construction of the figures to create generalizations, they can be encouraged to apply previously used strategies, like systematically rearranging the square tiles to create tile configurations that can be more easily generalized. Figure 20.7 illustrates how some teachers generalized the relationship between the tile number and the number of square tiles needed to construct the tile figure. As can be seen in figures 20.7a and 20.7b, teachers rearranged the tiles to create a square, ultimately leading to the generalization $(2n)^2 + 3$.

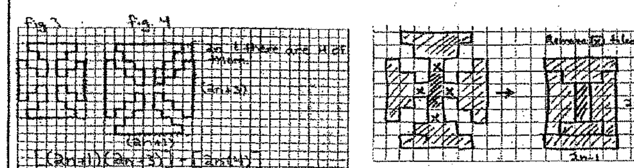
Two different ways teachers moved square tiles to create a large square, leading to the generalization:

# of square tiles needed to build the *n*th tile = $(2n)^2 + 3$



(a) Generalization 1: $(2n)^2 - 4 + 7 = (2n)^2 + 3$

(b) Generalization 2: $(2n)^2 + 3$



(c) Interpretation of Steve's Rule

(d) Interpretation of Esther's Rule

Fig. 20.7. Teachers' interpretation and creation of rules

One pair of teachers noticed that a "cross" made of seven tiles in the first figure is present in each of the subsequent figures in the pattern, and that identical arrangements of tiles can be found at the ends of each of the four "arms" of the cross. Since the cross remains fixed in each figure, they removed it and focused on the "changing aspect" of the figures, the configuration of the arms. They correctly observed that with each consecutive figure, an additional row of tiles, two tiles longer than the base row of the previous figure's arm, is added to the base of each arm. However, they could not coordinate the total number of tiles in the arm with the figure number. So, they color-coded each arm (to "see" that they had preserved the number of tiles in the figure) and rearranged the tiles to create a tile arrangement that can be indexed with the figure number and ultimately generalized. Eventually these two teachers found that by pushing the arms together, they could form a $2n \times 2n$ square that is missing four tiles in the center (see fig 20.7a). They then added seven tiles for the "cross" they had initially set aside, establishing the generalization of $(2n)^2 - 4 + 7$ tiles in the nth figure. Later they simplified the expression to $(2n)^2 + 3$.

Another pair of teachers also used a rearranging strategy to come up with this symbolic generalization. However, they partitioned the figure a bit differently. They separated each figure into a fixed center of three vertical tiles and four sets of "arms" (see fig. 20.7b). After color-coding, they rearranged the arms to create a $2n \times 2n$ square. Essentially, this is the same square the first pair found, except that since each of these arms has a tip consisting of one tile, the four tips come together to fill in the interior of this $2n \times 2n$ square. Also, since they removed the center three tiles, they found the relationship $(2n)^2 + 3$ directly without needing to simplify their symbolic expression.

After creating their own generalizations, the teachers were ready (though reluctant) to answer item 2, a "backwards" question that reverses the order in which they must analyze the representations of the relationship in the pattern. Now, the starting points for analyses were the symbolic rules of Pam, Esther, and Steve. Teachers had to construct tile designs that would generate the students' rules. To solve the problems, the teachers often applied strategies they had previously found helpful for creating a generalization, such as the rearranging strategy that is useful for interpreting Pam's and Esther's rules. For example, in figure 20.7d, one teacher partitioned the figure into four arms in the same way as the teachers did for figure 20.7a, but kept three tiles fixed in the center and removed four tiles from the "cross." In order to construct Esther's rule, this teacher rearranged the four arms and center three tiles to form a $(2n + 1) \times (2n - 1)$ rectangle with four extra tiles.

However, a rearranging strategy does not always work, and interpreting Steve's rule encourages a different type of analysis of the physical construction of tile design. As the teachers analyzed this symbolic generalization, they typically separated the figure into two parts: $[(2n + 1) \times (2n + 3)]$ and $(2n \times 4)$. Some teachers focused their attention on the expression $(2n + 1) \times (2n + 3)$ and looked to see if a rectangle with these dimensions could be formed in the figure. Others focused on the expression $2n \times 4$ and determined if the quantity 4 is subtracted (or "left blank") $2n$ times or if the quantity $2n$ is subtracted four times some place in the figure. Regardless of where they began their analyses, these teachers eventually saw the tile figure encased within a larger rectangle in which four groups of $2n$ tiles have been removed (see fig. 20.7c).

Using Tools at a More Abstract Level

Teachers can extend their understanding of generalization to a more abstract level by exploring tasks such as the Undoing-Formulas task (see fig. 20.8). Once

again, the typical order in which teachers make connections between the symbolic and pictorial representations is "backwards"; teachers start with a given symbolic representation of a relationship and then must physically construct a pictorial growth pattern that visually represents the rule. As part of this construction process, teachers must create and define variables and interpret how symbols can express a generalization of a relationship; they interpret change that is represented symbolically and then physically represent this change. It is imperative to encourage teachers to think of different ways to represent their rules pictorially so that the physical construction of the pattern clearly promotes the given symbolic rule. This approach of asking teachers backwards questions in which the teachers must engage in the construction process of creating a pictorial growth pattern not only deepens teachers' understanding of generalization and the flexible use of the tools but also encourages teachers to reanalyze the meanings of the operations and ways to represent them pictorially.

Here are four different "rules" for pictorial growth patterns.

Rule 1: $y = (3 \times n) + 2$

Rule 2: $y = [2 \times (n + 3)] - 1$

Rule 3: $y = [n \times (n + 1)] + (n - 1) + 2$

Rule 4: $y = (3 \times n) \times (n - 2)$

For each rule:

- Build at least three figures in a sequence that would lead to the rule. Make sure that the way the pattern is constructed lends itself to "seeing" this rule or generalization.
- Define or explain what n and y represent in your pattern.
- Write a description of the rule that would allow someone who has not seen your pattern or symbolic rule to re-create any figure in your pattern.

Fig. 20.8. Undoing-Formulas task

Initially, some teachers may tend to "plug in" values for n and draw that number of objects for each term in the sequence. Others strictly interpret the rules in terms of length, area, or volume and construct visual models accordingly. For example, in Rule 1, teachers may interpret $(3 \times n)$ as a rectangle with dimensions 3 by n, as 3 groups of n objects, or n groups of 3 objects (see fig. 20.9). Through an analysis of this rule, teachers can recognize that when a constant value is added or subtracted (in this instance, + 2), this part of the pattern should remain fixed from one figure to the next.

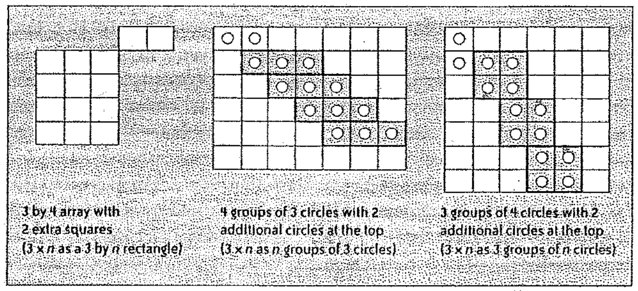


Fig. 20.9. Ways to represent $y = (3 \times n) + 2$ pictorially when $n = 4$

Pulling It All Together: Teachers' Change in Thinking

As teachers study pictorial growth patterns, they should explicitly be asked to reflect on the mathematical concepts and tools they have learned in each activity and how they extend or connect to concepts previously studied. After repeated experiences and intentional reflection, teachers may grow to appreciate how these different types of tasks build up essential components of algebraic thinking. After reflecting on her experiences, one teacher wrote:

> At the beginning of the semester, the patterns were something that I had fun doing, but [I] didn't really see the importance [in] doing them to develop algebraic thinking. Algebra, to me, was something that I did in 8th grade, not in elementary school, where I first learned about the patterns. Slowly, I began to see how extending patterns can lead to an algebraic representation of the pattern in a way that even elementary students could understand when looking at a pictorial representation rather then a numeric representation. By looking at the pictorial representations of the pattern, extending, and generalizing it based on geometric representation, students can begin to actually see why the generalization of the pattern works.

Every time I explore pictorial growth patterns with teachers, I marvel that one setting can provide such a rich context for supporting algebraic thinking. The activities presented in this article are designed to deepen teachers' abilities to generalize as they explore relationships between two different variables in pictorial growth patterns. Using the physical construction of a pattern, analyzing how the terms in a sequence change, and using symbols to represent and communicate the generalization become useful tools for creating, interpreting, and representing generalizations. Answering "backwards questions" provides additional opportunities for teachers to develop facility using and making connections among these tools.

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