STRENGTHENING PROSPECTIVE ELEMENTARY TEACHERS' UNDERSTANDING OF FACTORS

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Research on pre-service elementary school teachers' (PSTs') understanding of the multiplicative structure of number shows that PSTs struggle to use prime factorization to identify a number's factors. This study investigates the benefits of a sequence of three instructional tasks aimed at strengthening PSTs' understanding of factor by exploring the relation between a number's prime factorization and its factors. Analysis of written pre- and post-assessments of 69 pre- and in-service elementary and special education teachers shows that the use of these tasks strengthened PSTs' abilities to use prime factorization to identify factors and non-factors of both prime and composite numbers.

Consider the following question: Given the number $N = 2^3 \cdot 3^2 \cdot 5$, which of the following numbers $\{3,\ 7,\ 15,\ 22\}$ are factors of *N*? There are multiple routes to a solution. One approach is to compute the whole number value *N* and use trial division to determine if any of the factor candidates divide *N*. Assuming no computation errors are made, one will discover that 3 and 15 are factors of *N* and 7 and 22 are not factors of *N*. In addition to being both valid and straightforward, the approach is the most commonly learned method taught in schools.

A second approach is available that is more efficient, less prone to computational error, and richer in connections to underlying divisibility concepts. The approach relies on an important result in number theory – the Fundamental Theorem of Arithmetic (FTA). The theorem states that each natural number greater than 1 has a unique prime factorization (ignoring order). That is, no two natural numbers greater than 1 share the same prime factorization. Though seemingly benign, the notion of *uniqueness* is foundational to the field of number theory and provides the basis to the solution method. In this less frequently taught method, one examines the prime factorization of *N* to determine if it includes the prime factorization of 3, 7, 15, or 22. If it does include the prime factorization of any of these numbers, then that number is a factor of *N*. Otherwise, the given number is not a factor of *N*. Consider 3 in our example; since $N = 2^3 \cdot 3^2 \cdot 5 = (3) \cdot (2^3 \cdot 3 \cdot 5)$, *N* can be written as a multiple of 3 and so 3 is a factor of *N*. Similarly, 15 is a factor of *N* because the prime factorization of 15 is $15 = 3 \cdot 5$ and *N* can be expressed as $N = 2^3 \cdot 3^2 \cdot 5 = (3 \cdot 5) \cdot (2^3 \cdot 3)$, a multiple of 15. This approach can also determine non-factors. Since the prime factorization of *N* does not include the prime number 7 and no other prime factorization of *N* exists, 7 cannot be a factor of *N*. Likewise, since the prime factorization of 22 is $22 = 2 \cdot 11$, and since *N*'s prime factorization does not include *both* 2 and 11, we cannot write *N* as a multiple of 22. As a result, 22 is not a factor of *N*.

This introduction is meant to reveal the subtle complexities associated with the use of unique prime factorization guaranteed by the FTA to the analysis of factors. Using prime factorization requires knowledge of prime and composite numbers, the associative and commutative properties of multiplication, and a full understanding of the implications of the uniqueness feature of the FTA. In light of this fact, it should come as no surprise that many pre-service elementary school teachers (PSTs) struggle to make use of prime factorization and uniqueness to identify factors (Zazkis & Gadowsky, 2001). This documented difficulty points to PSTs' limited understandings of the multiplicative structure of the natural numbers. This study represents an attempt to remediate this area of concern by examining the benefits of a set of instructional tasks aimed at strengthening understanding of factors and prime factorization.

The Literature

Prior research has shown that pre-service elementary teachers (PSTs) struggle to understand divisibility concepts. The large majority of research in this area has been conducted by Rina Zazkis and her colleagues. In 1996, they conducted clinical interviews with 21 PSTs and found that 15 of them exhibited limited and procedural understandings of divisibility (Zazkis & Campbell, 1996a). When asked to find the factors of a number expressed in prime factored form, participants typically expressed the need to compute the whole number value of the number and perform long division. Zazkis (1998) corroborated this finding, showing that PSTs relied on empirical verification (e.g., long division or application of divisibility rules) and exhibited little ability to use prime factorization as a tool for reasoning about factors. Zazkis and Gadowsky (2001) framed this finding as PSTs' failure to make use of the *transparent* features of prime factorization. In other words, PSTs struggled to take advantage of the affordances that prime factored representations provide in terms of making certain numerical properties easily identifiable (e.g., the prime factored representation of $N = 2^3 \cdot 3^2 \cdot 5$ makes it transparent that 5 is a factor of *N*). Additional research by Brown et al. (2002) investigated how PSTs' divisibility schemas emerge and found that success using action-oriented strategies (e.g., trial division) may impede their use of prime factorization in completing divisibility tasks. They called for pedagogical interventions that emphasize flexible reasoning with numbers written in prime factored form.

Some studies have characterized the extent of PSTs' knowledge of number theory topics. Zazkis (2005) found that PSTs use negative descriptions to define prime numbers (e.g., "prime numbers 'cannot be divided', 'cannot be factored' or 'wouldn't have/are not having any other factor'" (p. 208)), which may be an obstacle to achieving a robust conceptual understanding of prime number. Other studies have identified PSTs' misconceptions about factors and prime numbers, such as the notion that bigger numbers have more factors or that prime numbers are small (Zazkis & Campbell, 1996b; Zazkis & Gadowsky, 2001). Researchers have also noted that PSTs tend to have an easier time identifying factors than non-factors, and are better able to recognize prime factors than composite factors (Zazkis & Campbell, 1996a, 1996b). Zazkis and Campbell (1996b) noted that PSTs' difficulty with identifying non-factors may be due to a lack of appreciation for the uniqueness feature of the FTA: "Whereas the *existence* of prime decomposition may be taken for granted, the *uniqueness* of prime decomposition appears to be counterintuitive and often a possibility of different prime decompositions is assumed" (p. 217).

Only a handful of studies have examined the efficacy of interventions aimed at improving PSTs' knowledge of divisibility and prime factorization. Feldman (2012) implemented a set of paper-and-pencil number theory tasks focused on the use of prime factorization with 59 pre-service elementary teachers. He found that their ability to identify factors and non-factors, as well as solve greatest common factor (GCF) and least common multiple (LCM) problems improved significantly. Also, Sinclair et al. (2004) and Liljedahl et al. (2006) used a computer applet that provided an interactive array where students could explore factors and multiples using a visual representational model in a laboratory setting. The researchers found that the combined effects of *visualization* and *experimentation* led to a more robust understanding of the multiplicative structure of the natural numbers, primes, composites, evens and odds.

Methodology

A classroom intervention was conducted at two large universities in the United States. Participants (n = 69) were undergraduate and graduate students enrolled in mathematics content courses for both pre-service and in-service elementary and special education teachers. Participants were asked to complete a pre-test before the start of the intervention and a post-test approximately two weeks after the conclusion of the intervention. The intervention, which lasted approximately three weeks, consisted of three in-class lessons and two out-of-class homework assignments. The goal of the intervention was for both instructors (the two authors) to facilitate participants' construction of their own understanding of prime factorization and divisibility concepts. In this role, each instructor observed participants' work in small groups of size three or four and only interrupted to ask guiding and probing questions. During whole-class discussions, each instructor encouraged participants to explain and justify their own mathematical thinking and rarely provided solutions themselves.

In Lesson 1, participants use factor trees and various factorizations of the same number to make sense of the Fundamental Theorem of Arithmetic and to recognize that although every counting number (greater than 1) has a unique *prime factorization* it may have several different *factorizations*. In Homework 1, participants fill a 10-by-10 array labeled from 1-100 with the prime factorization of each counting number using any method they choose. Participants then identify patterns in their array. In Lesson 2, participants use their arrays from Homework 1 to find factors of specific numbers in both whole number and prime-factored form (see **Figure 1** below). They discuss their conjectures for how a number's factors are related to its prime factorization. In Lesson 3, participants solve problems in order to develop a general rule for finding the number of factors using the number's prime factorization (i.e., $p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \cdot ... \cdot p_m^{n_m}$ has $(n_1+1)(n_2+1)(n_3+1)...(n_m+1)$ factors). Homework 2 is the final assignment of the intervention and serves to reinforce the discoveries made in Lessons 1-3.

Each participant completed written pre- and post-tests prior to and following the intervention. Three identical question types were used, but numbers were changed for pre- and post-tests in problems 1 and 2. Table 1 lists each test question. Across all questions, students were asked to show their work or provide their reasoning. Test scoring was conducted using a researcher-developed rubric. Inter-rater reliability analysis of the rubric was conducted. Both authors independently scored the same set of data (21.7% of the data, or 15 of 69 pre- and post-tests). Analysis of the results of the independent scoring revealed 82.5% agreement. Discrepancies in scoring were resolved via discussion and rubric clarification until 100% agreement was achieved. Once reliability had been established the remaining data were divided equally between the two authors and scored separately.

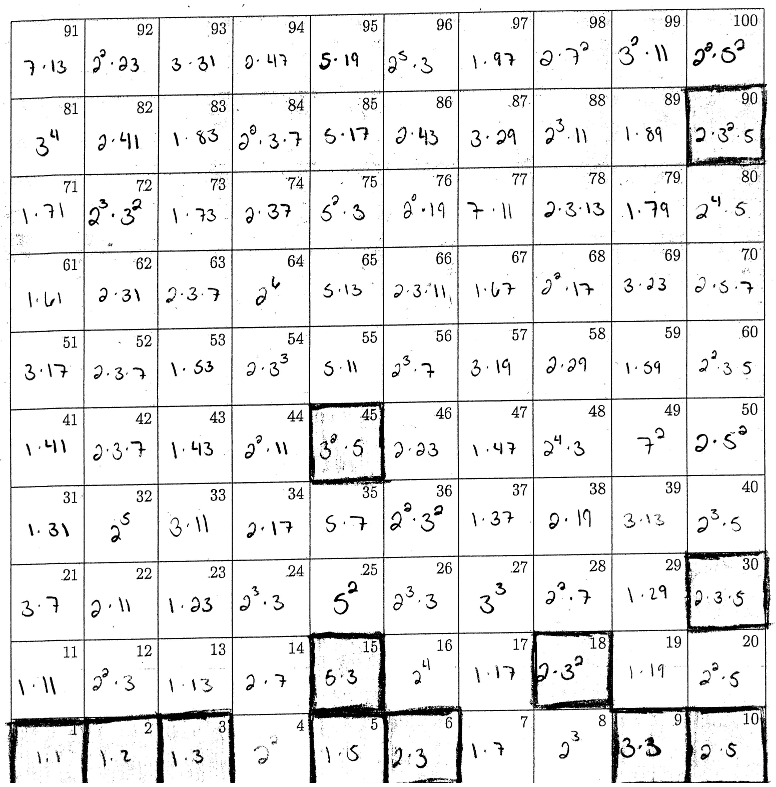

Figure 1: Example of participant work in Lesson 2

Table 1: *Description of Pre- and Post-Test Questions.*

Question	Prompt
1	Consider the number N = 3^2 x 5^4 x 11 x 17^3. Without calculating the value of N, determine whether each of the following is a factor of N. Justify each briefly. a) 5 b) 19 c) 15 d) 21 e) 75
2a	List all of the factors of 225. Show how you found all of them.
2b	List all of the factors of 5^2 x 7^2. Show how you found all of them.
3	What is the smallest positive integer that has the first ten counting numbers, 1 through 10, as its factors? Show or explain your work so that others can follow your logic. *Note: you may leave your answer in factored form.*

Findings

Analysis of the scoring data revealed that participants' abilities to solve number theory problems related to factors and prime factorization improved following the intervention described above. A paired sample *t*-test was conducted to compare participants' mean scores on the pre-test to their mean scores on the post-test. Results of the *t*-test indicated a significant difference between participants' pre-test (*M*=8.81, *SD*=4.40) and post-test scores (*M*=17.78, *SD*=4.97); $t(68)=-13.88$, $p< 0.05$. This suggests that participants' mean scores on the post-test were significantly greater than their mean scores on the pre-test. As such, the intervention supported participants' abilities to successfully solve problems related to factors and prime factorization.

Beyond this general result, the data provide information on how the intervention influenced participants' abilities to successfully identify prime factors, prime non-factors, composite factors, and composite non-factors. Prior research has shown that PSTs typically find it more challenging to identify non-factors than factors and composite factors than prime factors (Zazkis & Campbell, 1996a, 1996b). Table 2 shows mean scores, as a percent of available points on the scoring rubric, for Question 1a-1e on both pre- and post-tests. The table shows that participants improved in their ability to identify factors across all divisor types.

Table 2: *Mean scores for Question 1a-1e.*

Question	Divisor Type Given	Pre-Test	Post-Test
1a	Prime factor	62.3%	83.3%
1b	Prime non-factor	46.4%	81.2%
1c	Composite factor of form $p_1 \cdot p_2$	46.4%	78.3%
1d	Composite non-factor of form $p_1 \cdot p_2$	31.9%	70.3%
1e	Composite factor of form $p_1^2 \cdot p_2$	38.4%	75.4%

Interestingly, *differences* in PSTs' success rates on various types of problems appear to diminish following the intervention. Prior to the intervention, participants showed a marked difference in their ability to identify prime (62.3%) versus composite (46.4%) factors. Following the intervention, success rates in identifying prime (83.3%) and composite (78.3%) both increased while the difference between the two divisor types diminished. This result was maintained even when participants were faced with a more challenging composite factor as in question 1e. This reduction in performance differences following the intervention was also noted in participants' ability to identify factors versus non-factors. Prior to the intervention, participants were much more proficient at identifying prime factors (62.3%) than prime non-factors (46.4%) but were nearly equally proficient in these abilities following the intervention (83.3% and 81.2%, respectively). The result is similar in the case of composite numbers. Prior to the study, participants were more proficient at identifying composite factors (46.4%) than composite non-factors (31.9%). Following the intervention, success rates increased and differences diminished with a success rate of 78.3% in the case of a composite factor and 70.3% in the case of a composite non-factor.

Prior research shows that PSTs struggle to make use of a number's given prime factorization to identity its factors and instead revert to whole number conversion and long division. To assess the impact of the intervention on this area of concern, the study analyzed results obtained in questions 2a and 2b. In these questions, participants were asked to identify all of the factors of a particular number written in whole number (2a) and prime-factored (2b) forms (see Table 1). Participants were awarded credit in question 2a for finding all possible factors using any method. Participants were awarded credit in question 2b only if the participant made use of prime factorization in the construction of their response. Table 3 shows the mean scores, as a percent of available points on the scoring rubric, for both questions.

Table 3: *Mean scores for Question 2.*

Question	Pre-Test	Post-Test
2a	40.9%	72.5%
2b	25.5%	69.6%

The pre-test results replicate prior research by demonstrating that PSTs in the study exhibited greater difficulty in the identification of factors using prime-factored form (25.5%) as compared to whole number form (40.9%). Following the intervention, participants improved in their ability to find the factors of a number written in prime-factored (69.6%) and whole number form (72.5%) and, again, the difference in success rates in the two categories diminished substantially.

Conclusion

As the findings indicate, the number theory intervention is associated with a significant improvement in participants' abilities to solve problems related to factors and prime factorization. While the results of prior research appear to be validated by this study, the intervention shows promise in alleviating some of the challenges that PSTs have traditionally faced in learning these concepts. For instance, participants' ability to identify composite factors and non-factors improved substantially following the intervention, so much so that their success nearly equaled their success at finding prime factors and factors in general. One possible explanation for these improvements is that the intervention repeatedly asked participants to find composite factors and non-factors relying *only* on prime factorization. Increasing PSTs' exposure to prime factorization (Zazkis & Gadowsky, 2001) through activities incorporating visualization and exploration (i.e. Sinclair et al., 2004; Liljedahl et al., 2006) has shown promise in strengthening prospective teachers' understandings of factors. A larger, experimental study might confirm these results. The effect of teachers' different understandings of *factor* on students in their classrooms is also an area in need of greater investigation.

References

Brown, A, Thomas, K., & Tolias, G. (2002). Conceptions of divisibility: Success and understanding. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 41-82). Westport, CT: Ablex Publishing.

Feldman, Z. (2012). *Describing pre-service teachers' developing understanding of elementary number theory topics* (Doctoral dissertation, Boston University). Retrieved from http://gradworks.umi.com/35/29/3529017.html

Liljedahl, P., Sinclair, N., & Zazkis, R. (2006). Number concepts with *Number Worlds*: Thickening understandings. *International Journal of Mathematical Education in Science and Technology, 37(3)*, 253-275.

Sinclair, N., Zazkis, R., & Liljedahl, P. (2004). Number worlds: Visual and experimental access to elementary number theory concepts. *International Journal of Computers for Mathematical Learning, 8*, 235-263.

Zazkis, R. (1998). Odds and ends of odds and evens: An inquiry into students' understanding of even and odd numbers. *Educational Studies in Mathematics, 36,* 73-89.

Zazkis, R. (2005). Representing numbers: Prime and irrational. *International Journal of Mathematical Education in Science and Technology, 36,* 207-218.

Zazkis, R., & Campbell, S. R. (1996a). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education, 27(5),* 540-563.

Zazkis, R., & Campbell, S. R. (1996b). Prime decomposition: Understanding uniqueness. *Journal for Mathematical Behavior, 15,* 207-218.

Zazkis, R., & Gadowsky, K. (2001). Attending to transparent features of opaque representations of natural numbers. In A. A. Cuoco & F. R. Curcio (Eds.), *Roles of representation in school mathematics: 2001 yearbook* (pp. 44-52). Reston, VA: NCTM.