from Fostering Algebraic Thinking, by Mark Driscoll

6 Fostering Symbol Sense

Algebra and Symbolic Expression

As a subject, algebra defies any concise definition, largely because it has so many facets and uses. However, there is one thing that distinguishes it, even in the minds of those outside the education community: the use and manipulation of letter symbols. This impression is widespread for good reasons. Wheeler (1996) suggests that one of the "big ideas" related to algebra is "the idea, or the awareness, that an as-yet-unknown number, that a general number, and that a variable can each be symbolized and operated on 'as if' it was a number" (322).

There is a risk in the popularity of this impression about the role of symbolism in algebra, especially among teachers: that algebra and algebraic thinking become equated with the use of letter symbols. When this happens, use of letter symbols ceases to be a means to an end and becomes only an end, in and of itself; instead of using letter symbols to engage in higher order thinking such as generalizing and abstracting, people manipulate symbols as an end, in and of itself. We were reminded, during a session with a group of middle school teacher leaders involved in one of our projects, of the deep-seated mindsets of many teachers about the value of symbol manipulation. During this session, groups of teachers were working on Sums of Consecutive Numbers, the activity discussed in Chapters 2 and 4.

Sums of Consecutive Numbers

3 + 4 = 7

2 + 3 + 4 = 9

4 + 5 + 6 + 7 = 22

These problems are examples of sums of consecutive numbers. The number 7 is shown as the sum of two consecutive numbers. The number 9 is shown as the sum of three consecutive numbers. The number 22 is shown as the sum of four consecutive numbers. In this activity, you will explore what numbers can and cannot be made by sums of consecutive numbers.

The teachers had explored the mathematics in pairs and had been asked to think about where they had found themselves generalizing from calculation patterns they saw. Each pair reported back to the full group, using the overhead projector.

Doris, who taught in elementary school before joining the leadership team, reported on an investigation into numbers that can be represented as sums of four consecutive whole numbers:

"First we wrote them down in sequence:

1. $10 = 1 + 2 + 3 + 4$
2. $14 = 2 + 3 + 4 + 5$
3. $18 = 3 + 4 + 5 + 6$
4. $22 = 4 + 5 + 6 + 7$

$\vdots$

Then I looked at the right side of the equation signs and tried to capture what the pattern was in the computations. I noticed that you get the number on the left-hand side by taking the number of the equation, say, the *a*th equation, which is also the first number in the string of four on the right side. Add a to $a + 3$, the last number in the string, and multiply all times 2."

Doris then wrote "$2[a + (a + 3)]$" on the overhead and continued, "So, if you are in equation number 1, take $1 + (1 + 3)$, or 5 and multiply times 2 to get 10. In equation 2, $2 + (2 + 3)$, or 7, times 2 is 14, and so on." After brief acknowledgment that the formula worked, the group moved on.

As others talked about their investigations, someone used the algebraic expression $5n + 15$ to describe the numbers that can be represented as the sum of five consecutive whole numbers. Doris grimaced slightly but noticeably and, as soon as she had a chance to say something, came to the overhead projector and pointed to her expression, $2[a + (a + 3)]$. She said: "This is just simply $4a + 6$. That would be a simpler way to write it. I didn't need to write it out the long way." With that, she sat down, looking somewhat embarrassed.

Later, the facilitator realized that Doris' original expression, $2[a + (a + 3)]$, not only gave a clear rule for producing the sequence of numbers that can be written as the sum of four consecutive whole numbers, but also had embedded in it another generalization about numbers: that in any string of four consecutive integers, the sum of the first and last equals the sum of the middle two. In terms of giving insight into where Doris was generalizing from calculation patterns she saw, the topic of the group's discussion, this was an expression *preferable* to $4a + 6$.

There is a lesson here about the value of comparing and analyzing equivalent expressions, a point we return to later in this chapter. However, there also is a lesson here about the deep-seated mindsets regarding what has value in algebra—in this case, the most succinct symbolic representation. Doris' embarrassment was apparently rooted in a belief in the value of simplified expressions. However, in this instance, her original expression brought out her thinking and the underlying mathematics in a meaningful way, which would not have been possible had she simplified her expression.

Of course, there is great value in the succinctness and standardization of symbolic expression in algebra, and it is important that all students learn to use algebraic symbolism to express and communicate generalization, to reveal algebraic structure, to establish connections, and to formulate mathematical arguments. However, it is important to achieve these goals without pushing students prematurely and inappropriately toward using symbolic expression. Instead, it is important to build toward appropriate use of symbols by valuing the productive kinds of algebraic thinking that do not require symbolic expression.

Ideally, just as we want to foster, from the early grades, students' development of algebraic-thinking habits of mind, such as Building Rules to Represent Functions, Doing–Undoing, and Abstracting from Computation, we want them to develop good "symbol sense." In this chapter, we examine obstacles to learning how to understand and work with symbols, aspects of symbol sense that are important to heed, and implications for how to foster the development of symbol sense. We conclude with a sampling of activities that we feel can help spur the development of symbol sense.

Obstacles to Symbol Sense

Understanding the meaning of symbols and having a facility with symbols is difficult not only for algebra students, but also for many adults. We were reminded of the difficulties imposed by symbols one day during a summer institute for one of our projects.

On this day, we were concentrating on the mathematical habit of mind we call Doing–Undoing, the capacity to both build and take apart processes, or to investigate a process or relation by starting at the end and working backward (see Chapter 1). In this context, the teachers had been working in small groups on an activity, the beginning of which appears after this paragraph. In effect, the activity asks one to start with a division remainder and undo the process of division to arrive at starting conditions. The Age problem, which appeared in Chapter 2, is another version of this problem.

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1. Characterize the numbers for which you get a remainder of 1 when you divide by 5.
2. Characterize the numbers for which you get a remainder of 1 when you divide by 5, and a remainder of 2 when you divide by 3.

One of the small groups included Charles and Rhonda, both of whom were new to middle school and to the pressures of preparing middle school students to become proficient in algebra. For the second part of the problem, they wrote a string of numbers surrounded by numerous computations: 11, 26, 41, 56, 71, ... "We think this is all of them," said Charles. "But now what do we do with them?" asked Rhonda. The facilitator said, "Tell me what you see in that string of numbers," and, almost together, they said, "You start with 11 and go up by 15 each time." "But," Rhonda said, with some agitation, "there's gotta be more." Curious as to whether they saw any significance in the constant difference of 15, the facilitator asked, and learned that they had cranked out the first five numbers through computation and then had noticed the constant difference of 15.

Later, representatives from different groups came to the overhead projector to display what they had done and, in particular, to reveal which lines of their thinking "undid" the division process. During one of these presentations, a teacher was filling the transparency with a variety of references to what his group had done. In the course of this, he wrote $11 + 15n$ and said almost offhandedly that "this represents all the numbers that leave a remainder of 2 when divided by 3, and a remainder of 1 when divided by 5." From her seat, Rhonda said enthusiastically, "That's it! That's what we need to learn how to do." Presumably, Rhonda's "that" referred to forming an algebraic expression describing an infinite set of numbers—in this case, the arithmetic progression starting with 11 and progressing by constant differences of 15. Those who have been immersed in algebraic thinking for a long time can forget what a sizable leap it is to go from generating a finite string of numbers that are the same distance apart to having the symbol sense to form the concise symbolic expression that captures those numbers and all like them.

Rhonda and Charles' struggle is representative of difficulties many people have when using letter symbols. Working with letter symbols is challenging, in part, because of a set of fundamental obstacles that can get in the way of understanding the very *concept* of symbolic expression. Included in such obstacles are the following:

- **Differences between natural language and algebraic expression**: Tall and Thomas (1991) summarize this obstacle: "There is considerable cognitive conflict between the deeply ingrained implicit understanding of natural language and the symbolism of algebra" (125). One example offered is the expression $2 + 3x$, which is read left to right, but processed right to left, since the 3 and the x are multiplied before adding 2. Similarly, newcomers to algebraic expression can read the expression ab as a and b and so equate it with the expression $a + b$. These are two of the many ways in which the differences between natural language and algebraic expression can confound those who are trying to learn algebra.

- **Multiple meanings attached to letter symbols**: Letters in algebraic statements can refer to a single unknown number, as in $8n - 7 = 33$, but they also can refer to general numbers, as in the expression $8n - 7$, or to varying quantities, as in the function $n \rightarrow 8n - 7$. In prealgebra years, students often see elementary examples of letters that represent unknowns. It is often a challenge to interpret a switch in meaning. Even more complicated are statements in which the meaning is mixed. For example, textbooks often describe the general linear relationship by the formula $y = mx + b$. Here the y and x represent variables and the m and b represent "parameters." Arcavi (1994) points out that "the kinds of mathematical objects one obtains by substituting in them are very different" (30). In terms of the Cartesian plane, choosing numerical values for x and y fixes a point in the plane [the point $(x, mx + b)$]; choosing numerical values for m and b fixes a line in the plane [all the points (x,y) satisfying $y = mx + b$].

- **Cognitive difficulty in translating to algebraic expressions:** One of the central purposes of algebra is to model real situations mathematically, and a key to this is the capacity to translate from natural language to algebraic expression. This is the core challenge, for example, in algebra word problems. However, while some situations translate quite naturally (e.g., statements of equality, such as, "Express symbolically the relationship 'y is equal to the sum of x and 10'") others do not (e.g., statements comparing unequal quantities, such as, "Express symbolically the relationship 'y is 10 more than x'") (MacGregor & Stacey 1993). Algebraically, they are equivalent statements. Cognitively, however, they call for different mental representations.

  MacGregor and Stacey asked a group of several hundred ninth-graders to do the following tasks, each of which involves a comparison of unequal quantities:

1. "The number y is eight times the number z." Write this information in mathematical symbols.
2. "s and t are numbers. s is eight more than t." Write an equation showing the relation between s and t.
3. "The Niger river in Africa is y meters long. The Rhine in Europe is z meters long. The Niger is three times as long as the Rhine." Write an equation that shows how y is related to z.

For each of these three tasks, fewer than a third of the students were able to write a correct algebraic statement. Significantly, for each of the tasks, about half of the students wrote reversed equations such as $8y = z$ for the first one and $t = s + 8$ for the second. By contrast, the students did fine with this problem:

"z is equal to the sum of 3 and y." Write this information in mathematical symbols.

Therefore, it appears they were not misreading the statements of comparison of unequal quantities. Rather, the difficulty seems to be at the cognitive level. The authors state, "Students should be made aware that some relationships (such as 'eight more than') are easy to express in natural language and easy to comprehend but must be paraphrased, reorganized, or reinterpreted before they can be expressed mathematically" (229).

Facets of Symbol Sense

The kind of symbol sense that we recommend all students develop to support their algebraic thinking has several components (adapted from Arcavi 1994, Fey 1990), including those on the following list. This is not intended to be a comprehensive listing of what constitutes good symbol sense for algebra learners. However, the aspects of symbol sense listed serve as a helpful framework for this chapter for two reasons: (1) They address the heart of the teacher concerns that led to the writing of this chapter, and (2) they are closely related to the three algebraic-thinking habits of mind featured in this book: Doing–Undoing, Building Rules to Represent Functions, and Abstracting from Computation.

Knowing When to Call on Symbols and When Not To

Related algebraic-thinking guiding questions:

- How can I describe the steps without using specific inputs?
- Does my rule work for all cases?

It often makes sense to call on symbols when asked to show that some statement about numbers is "always true," as in, "What is always true about the difference between a whole number and its square?" ("Let's see. If n is the whole number, then $n^2 - n$ is the same as $n(n - 1)$, so one thing I can say is that the difference is the product of two consecutive numbers . . . .")

Sometimes the rush to "algebraicize" can make things more complicated mathematically than they need to be. Mason (1996) writes about how a group of students dealt with the following problem:

A number of eggs were brought to market. The number left remainders of 1, 2, 3, 4, 5, and 0 when put in groups of 2, 3, 4, 5, 6, and 7, respectively. What is the least number of eggs brought to market?

Mason observes, "The students rushed in to write down equations involving unknowns. . . . However, they were unable to do anything with their equations. This is an algorithm-seeking question, not a simple algebra question" (75).

Being Able to Interpret the Meaning of Symbols

In the previous section, we mentioned the multiple meanings that symbols can have in algebraic representation. For example, in $y = mx + b$, the description of the general linear relationship, the letters x and y have a different meaning than the letters m and b. Sometimes, even when only one letter occurs in an algebraic expression, it is important to look behind the symbol to interpret meaning. For example, in Chapter 5 we talked about the difficulty students can have in justifying a generalization such as, "The sum of two consecutive whole numbers is odd." In one study, many students were able to represent consecutive whole numbers with n and $n + 1$, and they were even able to express the sum as $2n + 1$; however, they were not able to see through the symbols in $2n + 1$ to conclude, "This has to be an odd number because $2n$ is a multiple of 2 and is always going to be even."

Related algebraic-thinking guiding questions:

- Is there information here that lets me predict what's going to happen?
- Now that I have an equation, how do the numbers (parameters) in the equation relate to the problem context?

Being Able to Inspect an Algebraic Operation and Predict the Form of the Result

An example of this aspect of symbol sense is knowing ahead of time that multiplying the expressions $(x + 6)$ and $(x^2 + 4)$ will result in an expression involving x^3 as the highest power of x. A more advanced example is seeing without computing that most terms will drop out if $(y - 1)$ is multiplied by $(1 + y + y^2 + \ldots + y^n)$.

Related algebraic-thinking guiding questions:

- How can I predict what's going to happen without doing all the calculation?
- What are my operation shortcuts for getting from here to there?

Knowing How to Scan an Expression or Formula and Make Rough Estimates of the Patterns That Would Emerge in Numeric or Graphical Representations

For example, you may recognize that if $4n + 5$ is represented on a graph for all natural numbers n, the points will all lie on a straight line.

Related algebraic-thinking guiding questions:

- How does this expression behave like that one?
- What are other ways to write this expression that will bring out hidden meaning?

- How can I write this expression in terms of things I care about?
- How are things changing?

Knowing How to Scan a Table or Graph, or to Interpret Verbally Stated Conditions to Identify the Likely Form of the Symbolic Expression of the Associated Algebraic Rule

This facet of symbol sense is, in essence, the reverse of the previous one. An example is recognizing (1) that if successive differences in a table of function values for inputs 1, 2, 3, . . . increase as an arithmetic progression, then the rule will include some constant times x^2; and (2) that this will be the highest power of x in the rule's formula. For example:

Input	*Function Values*	*Successive Differences*
1	5	
2	8	3
3	13	5
4	20	7
5	29	9
.	.	.

Related algebraic-thinking guiding questions:

- Is there information here that lets me predict what's going to happen?
- How are things changing?

Recommendations for Ongoing Attention to Symbol Sense

There are many ways in which teachers can support the development of symbol sense in their students on an everyday basis. They can start by both being aware of obstacles students face as they work with symbols and encouraging the development of various facets of symbol sense in their students. Following are some specific recommendations for developing symbol sense in students.

Use Visual Representation

To help smooth the transition to using letter symbols, give students plenty of opportunities to work with visual symbols that are not letter symbols. Often it is more natural to represent a quantity with a picture rather than a letter symbol. For example, when comparing prices of a cap and an umbrella, it can make more sense to represent the price of the cap with a picture of a cap and the price of an umbrella with a picture of an umbrella rather than to represent the prices with letter symbols. To help students make the transition to using letter symbols, it can help to first use pictorial symbols to represent quantities before moving on to the use of letter symbols. Finding Prices, an activity in the Example Activities section, requires that students represent prices of objects with pictures of objects. Chickens, another activity in the Example Activities section, also invites the use of pictorial symbols.

Capitalize on Opportunities

While it is important to be cautious about rushing students into using symbolic representation, it is equally important to look for opportunities to guide students toward using symbolic expression when their work shows signs of their being ready. For example, sometimes students will generate a handful of numerical examples that seem to reveal a consistent underlying process, which, in turn, lends itself to a smooth transition to symbolic expression.

Consider Figure 6–1, an example on the Sums of Consecutive Numbers activity. (This example is also discussed in Chapter 7, in another context.) Although the student falls well below standards for "describing the shortcuts" and "telling how," it is easy to infer that there is a well-reasoned procedure underlying the disconnected computations, one that seems based on and works backward from the recognition that three consecutive numbers beginning with n have the sum $3n + 3$; four consecutive numbers have the sum $4n + 6$; five have the sum $5n + 10$; and so on. This inference invites instructional questions such as, "Suppose you were given a number n that is divisible by 3. Could you express it as a sum of three consecutive numbers?"

FIGURE 6–1. *Student Work on Sums of Consecutive Numbers Activity*

4. Use the discoveries you made in question #2 to come up with shortcuts for writing the following numbers as the sum of two or more consecutive numbers. Describe the shortcuts you created and tell how you used them to write each of the numbers below as sums of consecutive numbers.

a) 45 b) 57 c) 62 d) 75 e) 80

$\frac{45-1}{2} = \frac{44}{2} = 22$

$22+23$

$\frac{45-3}{3} = \frac{42}{3} = 14$

$14+15+16$

$\frac{45-10}{5} = \frac{35}{5} = 7$

$7+8+9+10+11$

$\frac{45-15}{6} = \frac{30}{6} = 5$

$5+6+7+8+9+10$