TEACHING TEACHERS TO TEACH MATHEMATICS

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Group investigations from the areas of number theory, probability, and geometry, are presented and discussed. By working in groups, sharing ideas, and making and testing conjectures, prospective teachers gain confidence in their own ability to do mathematics and develop a variety of useful problem-solving strategies.

The National Council of Teachers of Mathematics Curriculum and Evaluation Standards for School Mathematics (1989), *Professional Standards for Teaching Mathematics* (1991) and *Assessment Standards for School Mathematics* (1995) establish problem solving, reasoning, and communicating mathematics as central concerns of mathematics education at all levels. The underlying assumption of these standards is that all students are capable of learning mathematics and that the mathematical development of each student in a diverse multicultural society must be valued (NCTM, 1995). One reason cited for the past failure of many students in mathematics is the overemphasis on rote memorization of facts and procedures rather than on underlying understandings. This has been equally true in collegiate mathematics courses and in the elementary classroom. As noted in the NCTM's *Professional Standards for Teaching Mathematics* (1991):

> How mathematics is taught is just as important as what is taught. Students' ability to reason, solve problems, and use mathematics to communicate will develop only if they are actively and frequently engaged in these processes.

If we expect this type of teaching in public schools, then college courses to train teachers must mirror the process. One way to involve all students in doing mathematics is to provide group investigations using manipulatives—physical objects that can be picked up, moved about, and rearranged to model mathematical ideas. Manipulatives have been used for many years to help students understand abstract ideas such as number, numeration, basic arithmetic operations, and spatial relationships. Support for the use of manipulatives in the primary grades dates back many years (Piaget, 1952, 1967; Dienes, 1964).

The rationale for the use of manipulatives in mathematics for students preparing to become early childhood and elementary school teachers is manyfold. One, for lasting changes in the teaching of mathematics in the early years to be made, the teaching of collegiate mathematics must also change (Mathematical Association of America, 1991). Prospective and practicing teachers need to experience adult-level insights into the mathematics they will or do teach, and to experience the same excitement in doing mathematics that they will and should foster in their own students (University of Chicago Conference, 1992). Two, we wish to model a learning environment that our students will feel comfortable using in their classrooms. Three, the adult learner, like the young learner, understands best when actively engaged in the construction or visualization of the mathematics. We have also found that when students work in groups, share ideas, and make and test conjectures, all students contribute and thus increase their confidence in their ability to do mathematics. In addition, anxieties and negative attitudes that have developed over a number of years decrease and sometimes even disappear.

This article presents investigations that have been successfully used in a mathematics problem-solving course, a probability course, a number-theory course, and an informal geometry course. Each investigation is followed by a brief discussion of possible outcomes, and one or two extension problems. Students work in pairs or groups of three or four on these investigations, thus providing a model for future classroom activities.

A problem-solving course emphasizes problem-solving strategies such as simplify the problem, gather and organize data, look for patterns, and generalize, if possible. The following two investigations with Cuisenaire© Rods incorporate these, as well as other problem-solving strategies.

INVESTIGATION: CUISENAIRE© ROD TRAINS

Problem **Suppose you have an unlimited supply of white and red Cuisenaire© Rods. Using just white and/or red rods, how many different trains can you make that match the orange rod? Cuisenaire©**

Rods are color-coded rectangular prisms sized from one centimeter to ten centimeters long. White rods are one cubic centimeter, red rods are the same length as two whites, and orange rods are ten whites long.

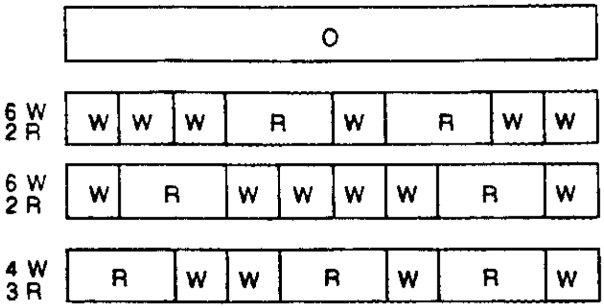

Figure 1: Different white/red trains that match an orange rod.

The purpose of this activity is to help preservice teachers to see the power of the "simplify the problem" strategy, which allows students to look for patterns and then make generalizations.

Discussion As students work on this problem by making trains, frustration generally sets in when they realize that there are many such trains. It is at this point that the meaning of "different" must be clarified, and a suggestion made to simplify the problem by investigating matching trains for each of the shorter rods: white, red, light green, purple, and yellow. Data should be organized in a table. While students work in small groups, results are checked with the total class. Out of this activity, the data shown in Table 1 is obtained.

Rod	Number of Whites Long	Number of Matching Trains With White and/or Red Rods
(W) White	1	1
(R) Red	2	2
(G) Light Green	3	3
(P) Purple	4	5
(Y) Yellow	5	8
(D) Dark Green	6	
(K) Black	7	
(N) Brown	8	
(E) Blue	9	
(O) Orange	10	

Table 1. Matching trains using white and/or red rods.

Patterns observed in the table are discussed, and guesses for the next entry in the table are made. These guesses are checked by making all the different trains that match the length of the dark green rod. At this point, some students have observed the pattern that each entry is found by adding the preceding two entries (a Fibonacci Sequence) and that the table can be extended to the orange rod. Students then realize why, with a total of 89 different red/white trains that match the orange rod, they experienced frustration in trying to make them all. The pattern arises from the activity, not from the numbers, per se.

The activity may be extended by giving an unlimited supply of white, red, and/or light green rods, and asking how many different trains can be made to match the orange rod. A further extension leads to Pascal's Triangle and powers of two. How many different trains can be made to match the orange rod if there are no restrictions on the colors of the cars?

Students are given a copy of Table 2. They are instructed to make matching trains for the shortest six rods. Students are directed this time to organize their procedure by looking at the number of one-car, two-car, and so forth trains that can be made to match each of the shortest six rods.

Rod	No. of Trains With These No. of Cars									Total No. of Trains
	1	2	3	4	5	6	7	8	9	10	
W	1										1
R	1	1									2
G	1	2	1								4
P											
Y											
D											
K											
N											
E											
O											

Table 2. Matching trains.

The whole class discusses the data, and patterns are observed and discussed. There are many patterns that may be observed, and students are made to feel good about any patterns they present. Some patterns that students observe include: (a) each row begins and ends with a 1, (b) each row of numbers is symmetric about the middle of the row, (c) the number of numbers in each row is one more than the row number, and (d) the total number of trains for each rod is a power of two.

Students with better mathematics backgrounds may observe that in organizing the data in this fashion, Pascal's Triangle is observed. As initial extensions to the data are conjectured, these are checked by using the rods. Finally, the table is extended using the noted patterns, and the problem question is answered: There are 512 different trains the length of the orange rod. Again, it is emphasized that the pattern is not in the numbers but in the activity from which the numbers are obtained.

The following activity presents preservice teachers with an opportunity to investigate the concept of probability.

INVESTIGATION: THE GAME OF WIN

Problem Construct this special set of dice.

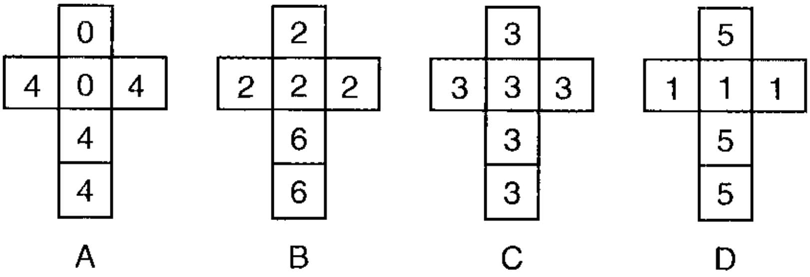

Figure 2: Dice faces for game of WIN.

This activity helps students experience a game of chance, understand the concept of sample space, and calculate and compare experimental and theoretical probabilities.

Play the game of WIN with a partner. Follow these rules.

- **Each of two players chooses one of the four dice.**
- **For each round, two players roll the dice simultaneously. The player with the larger number of dots on the top face wins the round.**
- **The game ends after 12 rounds.**

Play the game several times, taking turns to be the first to choose a die. Then answer these questions.

1. **Which die would you choose? Why?**
2. **Would you want to choose your die first or second? Why?**
3. **Suppose your opponent chooses die A and you choose die B. What is the sample space for this game? What is your probability of winning?**
4. **Suppose your opponent chooses die C. Should you choose die A or B or D? Why?**

5. **What is your strategy for playing WIN? If you follow your strategy, what is your probability of winning?**
6. **Play the game with your partner using your strategy. Comment on your strategy.**

Discussion This investigation is done after students have experience playing with standard dice and sample spaces. The game of WIN provides further experiences in describing sample spaces and calculating theoretical probabilities. The dice for this game of WIN were developed by Bradley Efron of Stanford University (Gardner, 1970). Initially, students assume that it is best to choose first. However, after playing the game, discussing sample spaces, and determining theoretical probabilities, they discover there is no one best die. In fact for each die there is another die that will "beat it" two times out of three. Figure 3 shows the sample space and probability for each pair of dice.

		B					
		2	2	2	2	6	6
	0	B	B	B	B	B	B
	0	B	B	B	B	B	B
A	4	A	A	A	A	B	B
	4	A	A	A	A	B	B
	4	A	A	A	A	B	B
	4	A	A	A	A	B	B

A vs B

P(A wins) = $\frac{16}{36} = \frac{4}{9}$

P(B wins) = $\frac{5}{9}$

		D					
		1	1	1	5	5	5
	0	D	D	D	D	D	D
	0	D	D	D	D	D	D
A	4	A	A	A	D	D	D
	4	A	A	A	D	D	D
	4	A	A	A	D	D	D
	4	A	A	A	D	D	D

A vs D

P(A wins) = $\frac{12}{36} = \frac{1}{3}$

P(B wins) = $\frac{2}{3}$

		D					
		1	1	1	5	5	5
	2	B	B	B	D	D	D
	2	B	B	B	D	D	D
B	2	B	B	B	D	D	D
	2	B	B	B	D	D	D
	6	B	B	B	B	B	B
	6	B	B	B	B	B	B

B vs D

P(A wins) = $\frac{24}{36} = \frac{2}{3}$

P(B wins) = $\frac{1}{3}$

		C
		3
	0	C
	0	C
A	4	A
	4	A
	4	A
	4	A

A vs C

P(A wins) = $\frac{2}{3}$

P(C wins) = $\frac{1}{3}$

		C
		3
	2	C
	2	C
B	2	C
	2	C
	6	B
	6	B

B vs C

P(B wins) = $\frac{1}{3}$

P(C wins) = $\frac{2}{3}$

		C
		3
	0	C
	0	C
D	4	C
	4	D
	4	D
	4	D

D vs C

P(D wins) = $\frac{1}{2}$

P(C wins) = $\frac{1}{2}$

Figure 3: Sample spaces and probabilities for game of WIN.

This investigation can be extended to include playing the game with three players. Students should consider questions such as: Would you want to choose your die first, second, or third? Why? What is your strategy for this game? What are the sample spaces and probabilities of winning?

The focus of the next activity is number theory and working backwards to develop a winning strategy, in order to observe patterns and then generalize the results.

INVESTIGATION: THE GAME OF NIM

Problem There are many versions of the game called NIM. One version was first played in China about 5,000 years ago. Students need chips or toothpicks for this game. Start with a pile of 30 chips. Two players take turns, each removing one, two, or three chips for their turn. The winner is the player who takes the last chip.

Variations Each player may pick up one, two, three, or four chips at each turn, or each player may pick up one, two, three, four, or five chips at each turn.

Students play NIM a few times with a partner and then answer questions such as: What is your strategy for winning? If you have a choice of whether or not to go first, tell what you would choose and why.

Discussion Since the end of the game determines the winner, one way to identify a winning strategy is to work backwards. What are winning and losing positions for this game? One chip is a winning position if it is your turn, since you can pick it up and win. Two chips is a winning position, since you can pick up two chips and win. And three chips is a winning position, since you can pick up all three chips and win. However, four chips is a losing position, since you cannot pick up all four chips, and you must pick up at least one chip, thus leaving behind one, two, or three chips which will be a winning position for your opponent.

One winning strategy would be to leave your opponent with four chips. How could you arrange to leave your opponent with exactly four chips? If your opponent had eight chips on the previous turn, then whether that opponent removed one, two, or three chips, you would be able to remove just enough to leave four chips remaining. So, you want to leave eight chips. How could that be arranged? By leaving 12 on the previous turn.

Continuing in this manner, work backwards to identify all the losing positions less than 30. These positions are: 4, 8, 12, 16, 20, 24, and 28. Notice these numbers are all multiples of four.

One strategy for winning is to go first and pick up two chips, leaving your opponent with 28 chips. Then, if your opponent takes one chip, you take three: if you opponent takes two chips, you take two chips, and if your opponent takes three chips, you take one chip. Notice that the sum of the number of chips you and your opponent remove together is always four.

If you are not able to go first, work to leave your opponent with 28, or 24, or some losing number of chips. Then continue with the strategy above.

How does the strategy change for the variations? If you are allowed to take one, two, three, or four chips, leave your opponent with some multiple of five chips. If you are allowed to take any number of chips from one to five, then the losing positions are multiples of six.

Further extensions of this game might ask students to find a winning strategy for playing the game starting with *n* chips where you can pick up any number from 1 to *m* chips at a turn. Or the rules might be changed so that the player taking the last chip is the loser. This changes the winning strategy in what way?

Geometry is one of the most difficult topics for beginning teachers to teach effectively. Investigations on geoboards help preservice teachers visualize squares and triangles in different positions, identify congruent figures, and determine areas of rectangles and triangles.

INVESTIGATION: AREA ON A GEOBOARD

Problem Define the one-unit length to be the distance between adjacent nails in a row or column on a five-pin by five-pin geoboard. Then the one square-unit area is the area of a square one unit on a side.

Try to construct squares with areas of one through ten or 16 square units. Which squares can you construct on your geoboard?

Exactly three of these squares are impossible to construct on a five-pin by five-pin geoboard. Identify these and explain why these squares cannot be made.

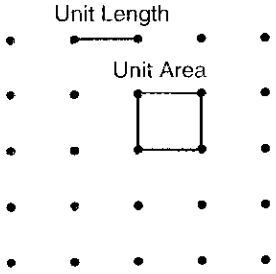

Figure 4: Unit length and unit area.

Discussion Students should be made aware that a five-pin by five-pin geoboard forms a square with sides that are four units in length. In general, an *n*-nail by *n*-nail geoboard forms a square with sides of length n-1. The eight noncongruent squares that can be constructed on a five-pin by five-pin board are shown in Figure 5.

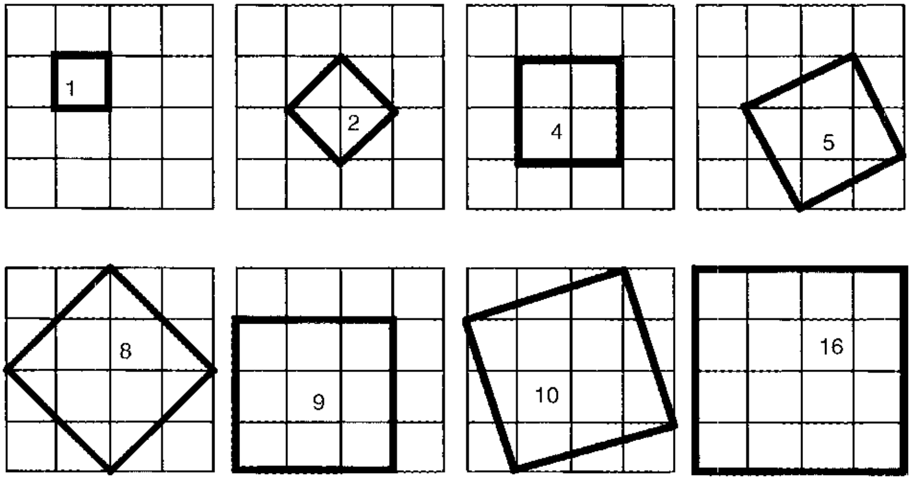

Figure 5: Squares on a 5-nail by 5-nail geoboard.

As shown in Figure 6, the area of a square can be determined by counting unit and half-unit squares, or by first determining the length of the side of the square. The length of the side of a square can be determined by finding a right triangle with the side of the square as the hypotenuse and using the Pythagorean Theorem.

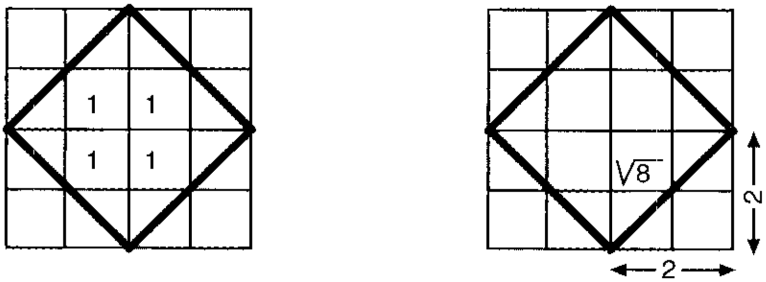

Figure 6: Two ways to determine area of a square.

It is impossible on square geoboards to construct squares of areas three, six, or seven square units. These would require sides of $\sqrt{3}$, $\sqrt{6}$, and $\sqrt{7}$, respectively. It is impossible to express these numbers as the sum of squares of whole numbers.

To extend this problem, consider squares that have sides parallel to the rows of nails normal squares. The rotated squares have sides that are not parallel to the rows of nails. How many noncongruent normal and non-congruent rotated squares can you make on a 2-nail by 2-nail geoboard? 3-nail by 3-nail geoboard? 4-nail by 4-nail geoboard? 5-nail by 5-nail geoboard? 6-nail by 6-nail geoboard? 7-nail by 7-nail geoboard? . . . n-nail by n-nail geoboard? Is it possible to have a rotated square equal in area to a normal square? If so, tell when this will happen. If not, why not?

Smaller geoboards can be made by partitioning off sections of the 5-nail by 5-nail geoboard. Larger geoboards can be made by joining four 5-nail by 5-nail geoboards, or by using square dot paper. The data for the different size geoboards is shown in Table 4.

Geoboard Dimensions	Number of Normal Squares	Number of Rotated Squares
2 by 2	1	0
3 by 3	2	1
4 by 4	3	2
5 by 5	4	4
6 by 6	5	6
7 by 7	6	9

Table 4: Number of normal and rotated squares.

The function rule for the normal squares is not difficult for most students to identify. The number of normal squares for an *n*-nail by *n*-nail geoboard is n-1. The function rule for the rotated squares is not easily identified. However, students may observe many patterns in the sequence of numbers, including the first differences of 1, 1, 2, 2, 3, 3, . . . and that the number of rotated squares for odd nail dimensions is a square number.

The first geoboard for which a normal and rotated square have the same area is an 8-nail by 8-nail board. The area of both squares is 25 square units each with side length of five units, as shown in Figure 7.

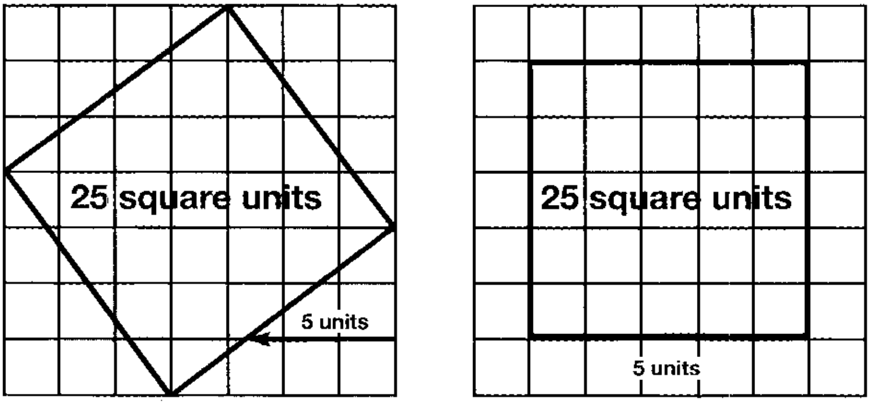

Figure 7: A rotated and normal square with area of 25 units.

Another extension of this problem asks students to construct triangles of area: ½, 1, 1½, 2, 2½, 3, 3½, 4, 4½, . . . , 8. After drawing the triangles on square dot paper, students explain how they determined the areas of the triangles, and determine the greatest triangle area that can be constructed on the geoboard.

There are 15 triangles of different areas that can be made on a 5-nail by 5-nail geoboard. There may be more than one way to form triangles with a given area. A triangle with an area of 8 square units is the largest triangle that can be made on a 5-nail by 5-nail geoboard.

CONCLUSION

It is a fair generalization to say that most teachers at all levels still talk too much! Lecturing may give the already talented student of mathematics a deeper understanding of the subject. Unfortunately, for many students, and surely for the prospective elementary school teacher, these usually well-organized and enthusiastic lectures fall on deaf ears. By using group investigations students are able to find their own methods, form and check (and even discard) their own conjectures, and experience the joy of "getting it by myself!" (with the help of other group members, of course). The instructor using group investigations must be wary of the temptation to give directions, even though some students may ask (even plead) for them. This is because they have not had the experience of exploring on their own, and, in fact, acting like real mathematicians.

There are many ways that students may communicate their frustrations, successes, methods, and outcomes of an investigation. Wheelock instructors in mathematics have successfully used group oral presentations with demonstrations, individual written reports, and student journal records. We have also found that when doing the investigations, our students have brought up questions that have led to still other investigations! In order to teach mathematics effectively so that all their young students tap their innate abilities, teachers of young children must experience the excitement of acting like mathematicians.

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