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Is It Always True? From Detecting Patterns to Forming Conjectures to Constructing Proofs

Suppose that a student comes to algebra class eager to share with her teacher and her classmates the following problem and its "amazing outcome." She leads the class through this set of steps:

> Choose any three-digit number whose leading digit is not zero.
> Double the number.
> Add 5.
> Multiply by 50.
> Add your age.
> Add 365.
> Subtract 615.

She then asks, "What do you notice about your new number?"

To each student's surprise, the result of performing this procedure is a five-digit number whose first three digits are the original three-digit number chosen and whose last two digits are the student's age. As students begin to realize that a similar result occurred for all students in the class, regardless of their initial choice of a three-digit number and the differences in their ages, the students' curiosity may be piqued and they may be motivated to investigate the mathematical ideas underlying this "amazing outcome." Students may begin to ask such important mathematical questions as, Does this outcome always occur? Why does this procedure produce a similar outcome regardless of the choice of three-digit number and age? Is it magic or is it mathematics at work?

Is it magic or mathematics?

REASONING AND PROOF IN ALGEBRA

Principles and Standards for School Mathematics (NCTM 2000) advocates that all students develop an understanding of the important roles that reasoning and proof play in mathematics. The Reasoning and Proof Standard suggests that students in the middle grades should "examine patterns and structures to detect regularities; formulate generalizations and conjectures about observed regularities; evaluate conjectures; and construct and evaluate mathematical arguments" (NCTM 2000, p. 262). It also indicates that high school students should have experiences with mathematical situations involving reasoning and proof that help them learn to "abstract and codify their observations" (NCTM 2000, p. 344) to prove that conjectures made from specific examples are true or to verify that something is impossible.

Situations such as the "amazing outcome" problem offer students opportunities to detect patterns, formulate conjectures, and construct simple algebraic proofs. First, within each student's solution, a relationship exists among the original three-digit number chosen, the student's age, and the five-digit number that is the result of executing this procedure. Then as students share their "amazing outcome" with one another, they discover that the same pattern identified within their specific case holds for all students in the class. From this observed pattern, students can formulate a conjecture that they can test by repeating the procedure with additional choices for the original three-digit number and the age. Accumulating many cases in which the conjecture is true gives students growing confidence that their conjecture is correct.

But questions left unanswered, even after students try many examples, are, Does this amazing outcome always occur? and Is our conjecture always true? Although students may generate many numerical cases, identifying and testing all possible cases are not feasible, even though a finite number of three-digit numbers and a limited number of reasonable ages can be used. And unless all possible

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cases are tested, the possibility that there is an exception to the rule or a case where the conjecture is not true always exists. Why does this amazing outcome occur? is an additional question that is open to further exploration. Through experiencing situations and attempting to answer questions such as these, algebra students come to recognize the need for and power of a general proof.

MOVING FROM CONJECTURES TO PROOFS

The "amazing outcome" problem involves the choice of two numbers: a three-digit number and an age. The conjecture formed is that the resulting five-digit number has a particular construction—the first three digits are the chosen three-digit number, and the last two digits are the student's age. Although students developed and confirmed this conjecture by using numerical examples, answering the question, Is it always true? requires a proof to ensure that this conjecture holds for all choices of three-digit numbers and ages. In addition, a proof has the potential to provide insight into the underlying mathematical structure of this "amazing outcome," revealing information that may have remained hidden or undetected when students worked only with specific numerical values. Thus, explaining the observed patterns and developing deeper mathematical understanding are positive outcomes of producing a proof (Knuth 2002).

Choosing appropriate algebraic notation is an important initial step in generating a proof of the conjecture. Instead of choosing specific numerical values for the three-digit number and the age, proving that the conjecture is true for all numbers requires employing variables to represent general cases. The use of a variable in this proof serves a different purpose than using a variable to represent an unknown numerical value for which an equation such as $2x + 3 = 15$ can be solved. In a proof, variables represent all elements in a particular set. The variable n here represents the elements of the set of three-digit numbers whose leading digit is not 0, and the variable a represents the elements of the set of two-digit numbers used to represent the age.

The second step in creating a proof of our conjecture is to apply each step in the procedure with the variables n and a. When possible, we simplify the algebraic expressions by using properties of addition and multiplication. The outcome, shown in **figure 1,** is an algebraic expression in terms of the two variables, n and a.

The outcome of this procedure, $100n + a$, shows that the original three-digit number, n, is multiplied by 100, creating a five-digit number with zeros as the last two digits. Adding a two-digit number, a, to this multiple of 100 yields a number of the expected form.

Choose any three-digit number.	n
Double the number.	$2n$
Add 5.	$2n + 5$
Multiply by 50.	$50(2n + 5) = 100n + 250$
Add your age.	$100n + 250 + a$
Add 365.	$100n + 250 + a + 365$ $= 100n + a + 615$
Subtract 615.	$100n + a + 615 - 615$ $= 100n + a$

Fig. 1

Explaining the observed patterns and developing mathematical understanding are positive outcomes of producing a proof

Once students identify the underlying mathematical structure and understand how the procedure has been constructed to achieve this amazing outcome, the problem situation can be extended in several ways. The teacher can challenge students to investigate whether similar results occur if the initial number chosen is a four-digit number or a two-digit number or how the number of digits in the age affects the outcome. Students can then develop appropriate notation and use the previous proof as a model for proving their new conjectures. In addition, students can create their own "amazing outcome" problems, in which they must pay close attention to the results of applying properties of addition and multiplication, the order of operations, and the relationship of the operations to one another. By engaging in such activities, students have multiple opportunities to increase and solidify their number sense.

CONJECTURES AND PROOFS INVOLVING CONSECUTIVE NUMBERS

Algebra students can learn to form conjectures and develop proofs to determine "Is it always true?" with different types of numbers (for example, odd and even numbers or consecutive integers) and different mathematical operations (addition, subtraction, multiplication, and division). Using consecutive integers to form, test, and prove conjectures is particularly appropriate for beginning algebra students, since the choice of notation is relatively easy to understand and since only basic algebraic skills are required to produce the proofs.

The three tasks outlined subsequently involve sums of consecutive positive integers. Each task allows students to create and investigate several numerical examples, form and test conjectures on the basis of these numerical examples, and prove their conjectures. Students can then look across the three tasks to form, test, and prove further conjectures. A powerful extension of these tasks is also provided; it will challenge and engage most high school algebra students. Martínez-Cruz and

Contreras (2002) present similar activities involving the products of consecutive integers.

Task 1:
What can be conjectured about the sum of any three consecutive positive integers?

Beginning algebra students can easily choose several sets of three consecutive positive integers and determine their sum.

$1 + 2 + 3 = 6$
$2 + 3 + 4 = 9$
$3 + 4 + 5 = 12$
$10 + 11 + 12 = 33$
$24 + 25 + 26 = 75$
$49 + 50 + 51 = 150$

What can we conjecture about the sum of n consecutive positive integers?

If students have previous experiences forming conjectures related to odd and even numbers, they first may look to see whether the sum of three consecutive positive integers is always odd or always even. As the preceding numerical examples show, the sum is odd at times and even at other times. Students can form a conjecture about the oddness or evenness of the sum of any three consecutive positive integers, but it must be stated using two distinct cases: the sum is even if the smallest (or largest) of the three consecutive positive integers is odd, and the sum is odd if the smallest (or largest) of the three consecutive positive integers is even.

In my experience, beginning algebra students quickly observe another pattern and conjecture that the sum of any three consecutive positive integers is a multiple of 3. This conjecture can be proved by choosing appropriate notation and representing the generalized sum, as follows:

Let p represent any positive integer.
The next two consecutive positive integers are represented by $p + 1$ and $p + 2$.
The sum of the three consecutive positive integers is $p + (p + 1) + (p + 2)$.
Simplifying the sum yields $3p + 3$.
Using the distributive property, rewrite the sum as $3(p + 1)$.
Since 3 is a factor of the sum of three consecutive positive integers, the sum is a multiple of 3.

The proof is concise but complete; we have determined that the conjecture is always true.

By examining numerical examples, students may form two additional conjectures, which can be explained by using the previous proof, about the sum of any three consecutive positive integers. First, the middle of the three consecutive positive integers, $p + 1$, is always a factor of the sum. Second, the sum of the three consecutive positive integers is always three times the middle number.

Task 2:
What can be conjectured about the sum of any five consecutive positive integers?

Once again, students may begin to explore this situation by creating numerical examples:

$1 + 2 + 3 + 4 + 5 = 15$
$2 + 3 + 4 + 5 + 6 = 20$
$8 + 9 + 10 + 11 + 12 = 50$
$25 + 26 + 27 + 28 + 29 = 135$

These numerical results lead to the conjecture that the sum of five consecutive positive integers is a multiple of 5. A proof, very similar to the preceding one, follows:

Let p represent any positive integer.
The next four consecutive positive integers are represented by $p + 1$, $p + 2$, $p + 3$, and $p + 4$.
The sum of the five consecutive positive integers is

$$p + (p + 1) + (p + 2) + (p + 3) + (p + 4).$$

Simplifying the sum yields $5p + 10$.
Using the distributive property, write the sum as $5(p + 2)$.
Since 5 is a factor of the sum of five consecutive positive integers, the sum is a multiple of 5.

Again, the middle number, $p + 2$, is also a factor of the sum; and the sum of five consecutive positive integers is five times the middle number.

Task 3:
What can be conjectured about the sum of any seven consecutive positive integers?

Noticing that task 3 has a structure that is similar to those of tasks 1 and 2, students may decide to form and prove a conjecture without creating any numerical examples. Stating that the sum of any seven consecutive positive integers is a multiple of 7 is a reasonable conjecture. Using the notation and structure of the previous proofs, we find that the conjecture is always true, since the sum is $7p + 21$, or $7(p + 3)$. Additional conjectures are that the middle number is a factor of the sum and that the sum is seven times the middle number. Again, the results of the proof show that these two additional conjectures are also true.

LOOKING ACROSS TASKS TO FORM NEW CONJECTURES

By now, a pattern that occurs across these three tasks emerges and serves as the basis for new conjectures. Thus far, we have proved that—

- the sum of three consecutive positive integers is a multiple of 3,

- the sum of five consecutive positive integers is a multiple of 5, and
- the sum of seven consecutive positive integers is a multiple of 7.

The observed pattern in these conjectures first leads us to the question, What can we conjecture about the sum of n consecutive positive integers? and then to the conjecture that the sum of n consecutive positive integers is a multiple of n. We must test and prove this conjecture to answer the question, "Is it always true?"

One way to begin testing this conjecture is to consider additional values of n. For $n = 4$, a numerical example, $1 + 2 + 3 + 4 = 10$, provides a counterexample, thereby proving that the conjecture is not true, since 10 is not a multiple of 4. Applying the notation and procedure illustrated in the previous tasks yields an explanation of why the sum of four consecutive positive integers is not a multiple of 4 for any choice of four consecutive positive integers (that is, the sum $4p + 6$ is not a multiple of 4 because 4 is not a factor of $4p + 6$). Similarly, the conjecture is not true for $n = 6$, since the sum $6p + 15$ is not a multiple of 6.

But students may notice another pattern that is based on the results of the previous investigation, and they may form a revised conjecture. Although the initial conjecture, the sum of n consecutive positive integers is a multiple of n, is not true for all n, it was proved true for $n = 3$, 5, and 7 and not true for $n = 4$ and $n = 6$. Two possible revised conjectures, each of which contains two cases, are—

1. The sum of n consecutive positive integers is a multiple of n when n is prime, and it is not a multiple of n when n is composite.
2. The sum of n consecutive positive integers is a multiple of n when n is odd, and it is not a multiple of n when n is even.

Investigating the case when $n = 9$ tests both these conjectures, since 9 is both a composite number and an odd number. The notation and procedure illustrated in the preceding tasks indicate that the sum of nine consecutive positive numbers is $9p + 36 = 9(p + 4)$. Since 9 is a factor of the sum, the sum is a multiple of 9. Therefore, we reject the first revised conjecture, whereas the second revised conjecture has passed the test, although it still has not been proved. Likewise, the case of $n = 2$, a prime number and an even number, can be used to test these two conjectures. Since the sum of any two consecutive numbers is odd and therefore not a multiple of 2, the first conjecture is again rejected, whereas the second conjecture appears to be true.

The proof of the second revised conjecture requires using two variables and may challenge beginning algebra students. But students with stronger algebraic skills should be able to participate in a teacher-led classroom discussion that results in creating the proof of this conjecture. The proof depends on representing the sum of the consecutive numbers from 1 to $(n - 1)$ as

$$\frac{n(n-1)}{2};$$

thus, teachers may want students to explore this relationship before they encounter it within the following proof:

Let n represent the number of consecutive positive integers.

Let p represent any positive integer. The additional consecutive positive integers are represented by

$$p + 1, p + 2, p + 3, \ldots, \quad p + (n - 3), p + (n - 2), p + (n - 1).$$

The sum of the n consecutive positive integers is

$$p + (p + 1) + (p + 2) + (p + 3) + \cdots + (p + (n - 3)) + (p + (n - 2)) + (p + (n - 1)).$$

Simplifying the sum of the n consecutive positive integers yields the expression

$$np + [1 + 2 + 3 + \cdots + (n - 3) + (n - 2) + (n - 1)].$$

Since the sum of the consecutive integers from 1 to $(n - 1)$ is

$$\frac{n(n-1)}{2},$$

the sum of n consecutive positive integers can be written as

$$np + \frac{n(n-1)}{2}.$$

Use the distributive property to rewrite the sum as

$$n\left[p + \frac{(n-1)}{2}\right].$$

If

$$\frac{(n-1)}{2}$$

is an integer, then n is a factor of the sum and the sum is a multiple of n. To determine whether

$$\frac{(n-1)}{2}$$

is an integer, consider two cases: when n is odd and when n is even.

Case 1: When n is odd, $(n - 1)$ is even, and therefore $(n - 1)$ is divisible by 2. So

$$\frac{(n-1)}{2}$$

is an integer when n is odd, n is a factor of the sum, and the sum is a multiple of n.

Case 2: When n is even, $(n - 1)$ is odd, and therefore $(n - 1)$ is not divisible by 2. So n is not a factor of the sum, and the sum is not a multiple of n when n is even.

Thus, we have completed a proof of the second revised conjecture.

CONCLUDING REMARKS

Students and teachers often associate the construction of proofs with the study of geometry and measurement. But opportunities abound for students to form conjectures and develop proofs in other mathematics content areas, as well. The problem situations presented in this article illustrate how algebra students can detect patterns and form conjectures that are based on their understanding of whole numbers and operations. In constructing algebraic proofs, students see the crucial connections between whole-number operations and algebra, learn the power of using a variable to represent all numbers of a particular type, perform generalized arithmetic, and arrive at conclusions from which they can proclaim with confidence and understanding, "Yes, it's always true."

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