Searching for Pythagorean Triples

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We all remember the Pythagorean theorem:

> For any right triangle, the sum of the squares of the sides adjacent to the right angle (the legs) is equal to the square of the side opposite the right angle (the hypotenuse).

In the traditional symbols, this can be expressed as $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse. When a, b, and c are all integers, they are said to form a Pythagorean triple. How many such triples are there, and how can they be found? Most students would know 3, 4, 5. Furthermore, multiplying these lengths by the same integer, one can generate infinitely many similar right triangles (6, 8, 10), (30, 40, 50), and so on.

Now, let's alter the original question a little bit: How many Pythagorean triples representing *nonsimilar* right triangles can an average middle school student name? Most likely, the answer is that very few triples can be named. Some students might remember 5, 12, 13. Memorizing many of these triples would be of limited value. What follows are activities that allow students to discover nonsimilar Pythagorean triples, with the aim of helping them gain insight into this immortal theorem.

Since the Pythagorean theorem operates with squares of numbers, we can use areas to represent them. Let's put the smaller square on top of the larger, as shown in **figure 1**. The shaded area represents the difference of two squares. If this difference itself is a square number, then we have a Pythagorean triple. Square tiles can be used to construct this design. This way, students can start simply by trying to arrange the tiles that represent the difference into a square. **Figure 2** shows the situation when the difference between the two squares (the green tiles) cannot be arranged into a square. In **figure 3**, the difference between the two squares (the green tiles) could form a square, thereby demonstrating the familiar Pythagorean triple: 3, 4, 5.

Fig. 1 When one square is superimposed on another, the difference of the two squares can be seen in the shaded area.

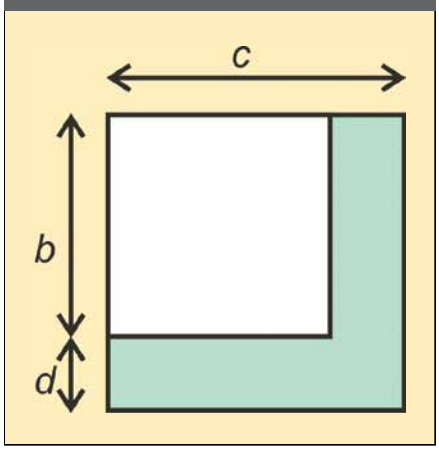

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Fig. 2 The area representing the difference of two squares, shown in green, may or may not be rearranged to generate a square. In the case of $4^2 - 3^2$, the remaining area of 7 square units cannot form a square.

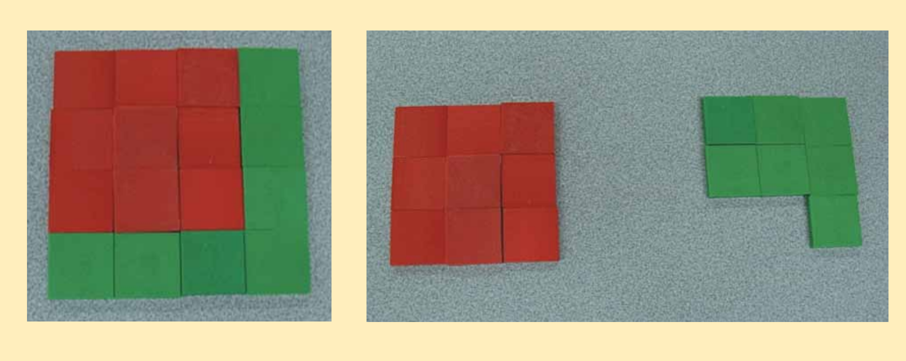


Fig. 3 In the case of $5^2 - 3^2$, the remaining area of 16 square units form a square with a side length of 4.

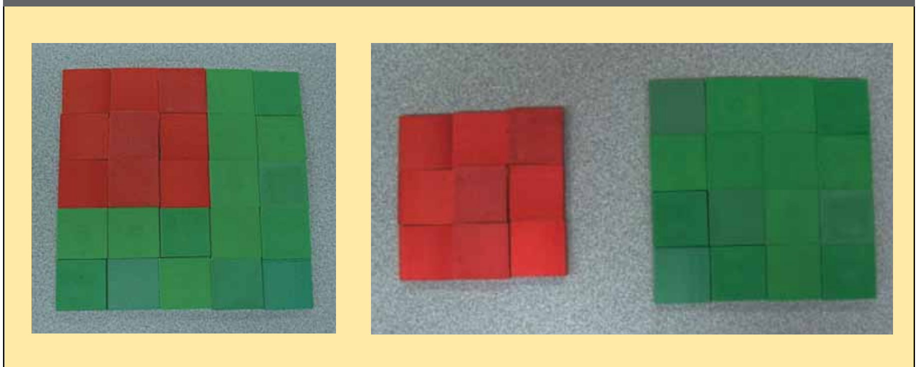


Fig. 4 When taking the difference of two squares, it is helpful to examine the cases where the difference between the side lengths of each square is 1; that is, where $b + 1 = c$ and the difference of the two squares will always be $2b + 1$.

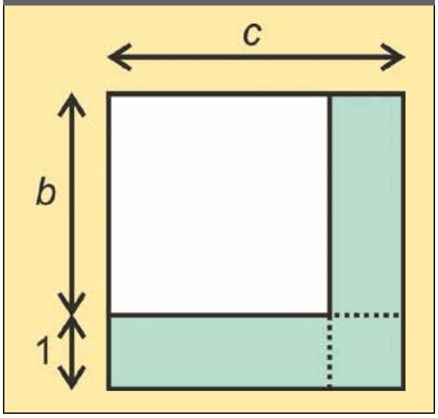
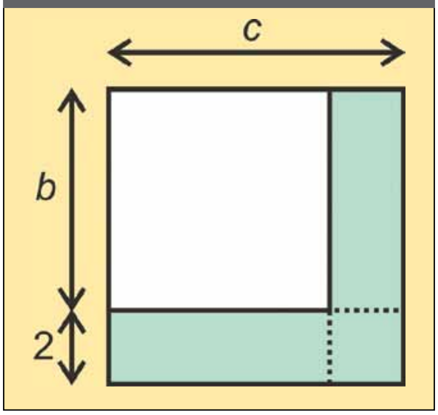


Fig. 5 When taking the difference of two squares, the cases where the difference between the side lengths is 2 generates additional Pythagorean triples.



SQUARES WITH A SIDE LENGTH DIFFERENCE OF 1

Using this approach, it may be convenient to classify the Pythagorean triples by the difference $(c - b)$. In other words, we will focus on a particular difference between the hypotenuse and one of the legs, which allows us to find the other leg. Let's start with the difference of 1. In this case, the area of the shaded region in **figure 1** can be written as $2b + 1$. **Figure 4** illustrates this particular case. Now, $2b + 1$ is a square number if and only if an integer x can be found such that $2b + 1 = x^2$, in which case

$$b = \frac{x^2 - 1}{2}.$$

This tells us that we will be looking for potential differences $(c^2 - b^2)$ only among odd square numbers, because if x^2 is even, b cannot be an integer. For example, if $x = 3$, $x^2 = 9$, then $b = 4$ and $c = 5$. From this, we see the familiar Pythagorean triple: 3, 4, 5.

Now, let's take $x = 5$. Then x^2 will equal 25, b will equal 12 from the formula, and c will equal 13 from the Pythagorean theorem. We therefore have another Pythagorean triple: 5, 12, 13. The next one can be found when $x = 7$, $x^2 = 49$, $b = 24$, and $c = 25$. For convenience, we can write them in a table. If generated by hand, we may find the first five rows in **table 1**. However, these values can be continued indefinitely using a spreadsheet to create any desired number of such triples.

Sometimes triangles represented by these triples are called *primitive right triangles*. A definition accessible to middle school students can be found, for example, in *An Adventurer's Guide to Number Theory* (Friedberg 1994). It is assumed that we are talking about sides that are integers:

> A right triangle is primitive if it is not the same shape as any smaller triangle. (p. 96)

Introducing this definition may lead to a helpful discussion on similarity of triangles. In light of finding Pythagorean triples, it can be emphasized that any triple given in **table 1** actually represents a whole class of Pythagorean triples. For example,

Table 1 Using the difference of two squares, where the smaller square has an odd length and the larger square is one unit greater, an infinite set of values can be found to generate nonsimilar Pythagorean triples.

x	x^2	$b = \frac{x^2 - 1}{2}$	$b + 1$
3	9	4	5
5	25	12	13
7	49	24	25
9	81	40	41
11	121	60	61
13	169	84	85
15	225	112	113
17	289	144	145
19	361	180	181
21	441	220	221
23	529	264	265
25	625	312	313
27	729	364	365
29	841	420	421
31	961	480	481
33	1089	544	545
35	1225	612	613
37	1369	684	685
39	1521	760	761
41	1681	840	841
43	1849	924	925
45	2025	1012	1013
…	…	…	…

Table 2 These numbers result when generating Pythagorean triples in which the difference of two squares is 2.

x	x^2	$b = \frac{x^2 - 4}{4}$	$c = b + 2$
4	16	3	5
6	36	8	10
8	64	15	17
10	100	24	26
12	144	35	37
…	…	…	…

5, 12, 13 may be used to produce any triple that can be written in the form $5k$, $12k$, and $13k$, where k is the same positive integer. Thus, 10, 24, 26; 25, 60, 65; and 50, 120, 130 are Pythagorean triples, as well.

SQUARES WITH A SIDE LENGTH DIFFERENCE OF 2

Let's consider the case where the square side lengths have a difference of 2. In this case, the area of the shaded region in **figure 1** can be written as $4b + 4$. (See **fig. 5**.) Now $4b + 4$ is a square number if and only if an integer x can be found such that $4b + 4 = x^2$, in which case

$$b = \frac{x^2 - 4}{4}.$$

This representation tells us that we will be looking for potential differences ($c^2 - b^2$) only among even square numbers, because if x^2 is odd, b cannot be an integer. For example, if $x = 4$, $x^2 = 16$, then $b = 3$ from the formula above and $c = 5$ from the Pythagorean theorem. Again, we have the familiar 3, 4, 5. If $x = 6$, then $x^2 = 36$, $b = 8$, and $c = 10$. A triangle with these sides is similar to the triangle with the sides 3, 4, 5. The next even number (8) produces 8, 15, 17, which is truly a "new triple."

For convenience, we can write the triples found into **table 2**. This table shows the first five odd squares, and these numbers can be continued indefinitely. Every even line in this table represents a triangle that is similar (with coefficient 2) to a triangle represented in **table 1**. For example, the fourth line shows

$$x = 10,\ x^2 = 100,\ b = 24, \text{ and } c = 26.$$

Thus, we have a right triangle with sides 10, 24, and 26. Every side of this triangle is twice as long as the corresponding side of the triangle 5, 12, 13, which can be found in the second line of the **table 1**.

This method can be used to create Pythagorean triples with any desired difference between a hypotenuse and a leg.

PAIRS OF SQUARES WITH OTHER SIDE LENGTH DIFFERENCES

At this point, it may be interesting to consider possible generalizations of the numerical information collected in **tables 1** and **2**. Let's start with **table 1**. Every x in this table can be written as $2n + 1$, where n is a positive integer. Thus,

$$x^2 = 4n^2 + 4n + 1.$$

Consequently,

$$b = 2n^2 + 2n \text{ and } c = 2n^2 + 2n + 1.$$

Therefore, every Pythagorean triple with the difference of 1 between the two greatest numbers can be found by substituting an integer n into these three formulas. For example if $n = 17$, we have 35, 612, 613.

Similarly, analyzing **table 2**, it can be noticed that x can be written as $2(n + 1)$, where n is a positive integer. Thus,

$$x^2 = 4n^2 + 8n + 4.$$

Consequently,

$$b = n^2 + 2n \text{ and } c = n^2 + 2n + 2.$$

Therefore, every Pythagorean triple with the difference of 2 between the two greatest numbers can be found by substituting an integer n into these three formulas. For example if $n = 17$, we have 36, 323, 325.

The method of discovering Pythagorean triples described here is relatively simple. At the same time, it demonstrates for middle-grades students the power of a systematic approach to a solution. Although it may be rather difficult to find three numbers with the given property by adding up squares of randomly selected integers in the hope of getting a square number (through a trial-and-error method), the suggested approach easily produces infinitely many triples. In addition, this method emphasizes the strong connections between geometric and algebraic representations. This may help middle-grades students see mathematics in general, as a well-structured system of interconnected ideas.

REFERENCES

Friedberg, Richard. 1994. *An Adventurer's Guide to Number Theory.* New York: Dover Publications.

Maor, Eli. 2007. *The Pythagorean Theorem: A 4,000 Year History.* Princeton, NJ: Princeton University Press.

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Did You Know?

Although the theorem is named after Pythagoras (approximately 500 BC), it is interesting to note that it seems to have been known in Mesopotamia more than a thousand years earlier. The details of this fascinating story can be found in *The Pythagorean Theorem: A 4,000 Year History* by Eli Maor (2007).

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