DELVING deeper

Jeffrey Steckroth

A Transformational Approach to Slip-Slide Factoring

The slip-slide method of factoring first came to my attention in the late 1980s when I was tutoring a student who was taking Algebra 1 as a seventh grader. George was working on factoring quadratic trinomials and shared with me the unusual technique that he had learned at school. He showed me this example:

$$\begin{gathered} 6x^2 + 5x - 4 \\ x^2 + 5x - 24 \\ (x + 8)(x - 3) \\ (x + 8/6)(x - 3/6) \\ (x + 4/3)(x - 1/2) \\ (3x + 4)(2x - 1) \end{gathered}$$

George demonstrated the method to me several times with other quadratic trinomials, and each time the method produced the correct pair of factors. I was stunned.

I was fascinated by the method and, at the same time, puzzled. I could find no counterexample to show that the method did not always work, but I was unable to demonstrate mathematically that the method *would* always work. This bothered me. As a teacher, I was proud of the fact that I always made sure that my students knew why a method or procedure worked, yet now I had a method that I, the teacher, could not explain.

Before reading further, you may wish to consider the steps shown above and see if you can make sense of what is going on.

WHAT IS THE SLIP-SLIDE METHOD?

For readers who are not familiar with the slip-slide method of factoring quadratic trinomials, let's take a closer look at the example $6x^2 + 5x - 4$. Traditionally, the approach to factoring has been to use trial and error and consider all the possible pairs of linear binomials that could be created until one was found that produced the correct middle term. In this case, the leading term $6x^2$ has two possible factor pairs: $(6x)(x)$ and $(3x)(2x)$. The constant -4 has three pairs of factors: $(-4)(1)$, $(4)(-1)$, and $(2)(-2)$. For this example, then, the possible pairs of binomial factors would include those found in **table 1**.

The existence of common factors in some of the potential binomial factors would allow you to eliminate many of these without having to determine their middle terms. Even so, the process is tedious, particularly with trinomials whose leading coefficient and constant have multiple factors. Once you found the pair of binomial factors that produced the correct middle term, you were done. Here, we find that the factors of $6x^2 + 5x - 4$ are $(3x + 4)$ and $(2x - 1)$.

Slip-slide factoring eliminates the need to create a long list of potential factors or resort to trial and error. Instead, a predictable sequence of steps produces the correct factors of any nonprime quadratic trinomial with integer coefficients a, b, and c. If you were using the slip-slide method, you would follow the steps shown in **table 2**.

If you were seeing this for the first time, three steps in the process would likely bother you: steps

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Table 1 Binomial Pairs Related to the Quadratic Trinomial $6x^2 + 5x - 4$

Factors	Middle Term
$(6x + 1)(x - 4)$	$-23x$
$(6x - 4)(x + 1)$	$2x$
$(6x - 1)(x + 4)$	$23x$
$(6x + 4)(x - 1)$	$-2x$
$(6x + 2)(x - 2)$	$-10x$
$(6x - 2)(x + 2)$	$10x$
$(3x + 1)(2x - 4)$	$-10x$
$(3x - 4)(2x + 1)$	$-5x$
$(3x - 1)(2x + 4)$	$10x$
$(3x + 4)(2x - 1)$	**$5x$**
$(3x + 2)(2x - 2)$	$-2x$
$(3x - 2)(2x + 2)$	$2x$

1, 3, and 5. In step 1, why would you multiply the leading coefficient and the constant of the original trinomial? The new trinomial that you create is not equivalent to the one that you started with. That bothered me. Next, in step 3, what gives you the right to arbitrarily divide each constant by the original leading coefficient? Doing so changes the two binomial factors. That bothered me. Finally, after simplifying the fractions in step 4, why would you "slide" the denominators of the fractions and make them the coefficients of x in step 5? That really bothered me.

I could rationalize that steps 1 and 3 somehow made sense because multiplication and division were inverse operations. Multiplying by 6 and dividing by 6 in some way seemed okay, yet I could not find a mathematical way to justify those maneuvers. That issue aside, I was truly bewildered by step 5, in which we simply slide the denominators to the left, magically turning fractions into whole numbers and creating two linear binomials that were always the correct factors of the original quadratic trinomial.

This method of factoring reminds me of my introduction to finding square roots by hand, a method I learned from my father long before the arrival of handheld calculators. He showed me a tedious series of steps that enabled me to calculate square roots to any degree of precision, depending on my level of patience. I did not understand why the procedure worked, but it did, and I continue to amaze people today with my remarkable ability to extract square roots by hand.

Slip-slide factoring resembles finding square roots by hand—the method works, but why it works is unclear to most people, including teachers. Students can learn to mimic the procedure successfully, but they do so without understanding, a situation I find untenable.

The authors of *Adding It Up* (NRC 2001) discuss at length the five strands of mathematical proficiency, two of which are procedural competence and conceptual understanding. *Procedural competence* focuses on the ability to perform procedures correctly to arrive at the desired result, similar to the manner in which one might extract a square root by hand. *Conceptual understanding* has to do with whether one understands why a procedure works—for example, why can one simply "move" the decimal point to the right or left when multiplying or dividing by a power of ten. *Adding It Up* makes the case that the acquisition of one strand of proficiency without the other is inadequate to ensure overall mathematical proficiency in our students. For students to attain a high level of mathematical proficiency, they must have procedural competence, but they must also possess conceptual understanding.

Table 2 The Slip-Slide Method of Factoring

Step	Objective: Factor the quadratic trinomial shown	$6x^2 + 5x - 4$
1	Multiply the coefficient of x^2 by the value of the constant term and let this be the new constant. Drop the coefficient of x^2. (That is, replace the coefficient with 1.)	$x^2 + 5x - 24$
2	Factor the new trinomial.	$(x + 8)(x - 3)$
3	Divide the constant in each binomial factor by the original coefficient of x^2.	$(x + 8/6)(x - 3/6)$
4	Simplify resulting fractions, if possible.	$(x + 4/3)(x - 1/2)$
5	In each binomial factor, if the constant is a fraction, make the denominator of the fraction the coefficient of x and let the numerator be the new constant.	$(3x + 4)(2x - 1)$

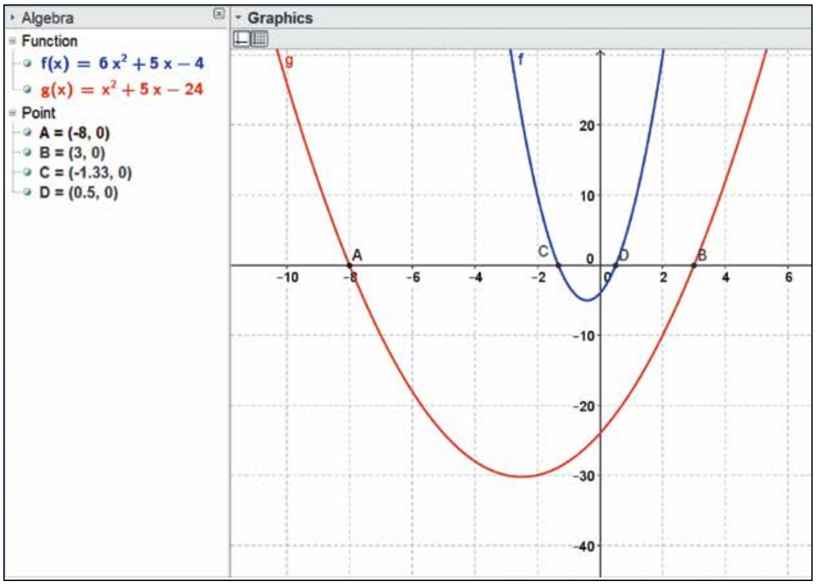

Fig. 1 Graphs show zeros of the primary function $f(x)$ and of the secondary function $g(x)$.

Table 3 Finding Zeros of the Functions Algebraically

$f(x) = 6x^2 + 5x - 4$	$g(x) = x^2 + 5x - 24$
$0 = 6x^2 + 5x - 4$ $= (3x + 4)(2x - 1)$	$0 = x^2 + 5x - 24$ $= (x + 8)(x - 3)$
$x = -4/3$ or $x = 1/2$	$x = -8$ or $x = 3$

Whether the skill involves finding square roots by hand, multiplying by powers of ten, or factoring quadratic trinomials, performing the procedure alone without an understanding of why the procedure is valid leaves students with an incomplete knowledge base from which to operate; their competence in performing a particular procedure has no connection to prior knowledge. NCTM's Connections Standard (NCTM 2000, p. 64) suggests that since such "learning" may be temporary, all learning should be built on prior knowledge for it to have meaning.

As the years passed, I did teach this method of factoring to my students and gave them some feeble explanation about why it worked. Secretly, I knew that I was being unfair to my students and untrue to my own principles. Time and again, I would try to justify the slip-slide method to myself algebraically, and each time I would wind up frustrated and more discouraged. Periodically I would revisit the method, but the results were always the same.

A GRAPHICAL EPIPHANY

More than twenty years after first learning about the slip-slide method of factoring and after numerous failed attempts to justify the method to my own satisfaction, I began to explore slip-sliding anew. This time, however, I approached the task from a different angle. Rather than considering only the algebraic approach, I decided to look at the task graphically. Tools such as a graphing calculator and interactive geometry software enabled me to consider visual representations that had not been readily available two decades earlier. Today's technology made the process effortless, accurate, and quick. It now seemed natural to investigate graphically, whereas decades ago this was not an option.

The slip-slide process began to make sense when I finally accepted the fact that the polynomial produced in each step was not intended to be equivalent to the polynomial in the preceding step. The issue is not equality among polynomial expressions; the issue is how each polynomial relates to the others. Instead of worrying about the fact that each expression was not mathematically equivalent to the others, I began to treat each polynomial as a polynomial function. I started to look at the graphical representations of the various polynomial functions to see what effect each modification had on the graph of the preceding function. This approach allowed me to look at the process in an entirely new light and make connections between the characteristics of polynomial functions and the factors of polynomial expressions.

This transition from *polynomial expression* to *polynomial function* is one that should not be taken lightly. We are moving from an expression in one variable to a functional relationship in the form of an equation in two variables. Although teachers routinely move back and forth from one to the other, using them almost interchangeably, this transition is a major conceptual barrier for students who are just beginning to grasp the notion of function. We would serve our students well if we could do a better job of helping them appreciate the differences and similarities between the two. (But that is a topic for another time.)

I decided to examine the relationship between the graphs of the two associated quadratic functions, $f(x) = 6x^2 + 5x - 4$ and $g(x) = x^2 + 5x - 24$, that appeared in the first two lines of the slip-slide example in **table 2**. I hoped to notice a relationship between the two functions that would shed light on why the slip-slide method works.

FACTORS AND ZEROS

Beginning algebra students ordinarily learn how to factor quadratic trinomials before learning about the characteristics of quadratic functions. They first learn the mechanics of factoring, after which they learn to solve quadratic equations by

Table 4 Finding Vertices of the Functions Algebraically

$f(x) = 6x^2 + 5x - 4$		$g(x) = x^2 + 5x - 24$	
$x = -\frac{b}{2a}$ $= -\frac{5}{2 \cdot 6}$ $= -\frac{5}{12}$	$y = f\left(-\frac{5}{12}\right)$ $= 6\left(-\frac{5}{12}\right)^2 + 5\left(-\frac{5}{12}\right) - 4$ $= -\frac{121}{24}$	$x = -\frac{b}{2a}$ $= -\frac{5}{2 \cdot 1}$ $= -\frac{5}{2}$	$y = g\left(-\frac{5}{2}\right)$ $= \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) - 24$ $= -\frac{121}{4}$
$V_f = \left(-\frac{5}{12}, -\frac{121}{24}\right)$		$V_g = \left(-\frac{5}{2}, -\frac{121}{4}\right)$	

factoring or by using the quadratic formula. Only later are they asked to apply these techniques to find the zeros of quadratic functions. As a result, students may not make the connection between factors and zeros until long after they have mastered the art of factoring. Those who are able to connect the dots between factors and zeros, however, understand that those two concepts are closely related.

Unable to make the connection between factors of two associated polynomial expressions algebraically, I turned my attention to the related polynomial functions $f(x)$ and $g(x)$ and their graphs. This was a game changer. What follows is an account of how I was able to compare the two functions graphically, numerically, and algebraically and begin to make sense of why the slip-slide method of factoring works.

THE INVESTIGATION

The trinomial I wanted to factor, the primary quadratic, was $6x^2 + 5x - 4$, so I created a related primary polynomial function, $f(x) = 6x^2 + 5x - 4$, and graphed it using GeoGebra. Then I took the secondary polynomial, $x^2 + 5x - 24$, and created a related secondary function, $g(x) = x^2 + 5x - 24$, adding its graph to the same GeoGebra sketch.

The graphs of the two polynomial functions were what you would expect. The primary function, $f(x)$, had a more narrow shape than $g(x)$ because of its greater leading coefficient. The secondary polynomial function, $g(x)$, was much lower on the graph than the first, given its smaller constant term. Both parabolas opened upward because the leading coefficients were positive.

A look at the graphs of the two functions also revealed that the zeros of $f(x)$ were relatively close to the origin, whereas the zeros of $g(x)$ were significantly farther from the origin. I first used GeoGebra to find the zeros (see **fig. 1**) and then found them algebraically, as shown in **table 3**. If you look carefully at the two pairs of zeros, you may notice a relationship between them.

I did not immediately see a relationship between the two pairs of zeros and decided to investigate the vertices of the two parabolas as well. From the graph, it appeared that the vertex of $f(x)$ was much closer to the origin than the vertex of $g(x)$, and the calculations confirmed this. The algebraic process of finding the vertex coordinates of the two parabolas is shown in **table 4**.

I could not help but notice that the coordinates of the vertices of the two functions were remarkably similar. Both ordered pairs had the same numerators, and the denominators of V_f were six times as large as the corresponding denominators of V_g.

That ratio of 6 : 1 caught my eye, since the ratio of the leading coefficients of the two functions, $f(x)$ and $g(x)$, was also 6 : 1. I decided to take another look at the zeros of the two functions to see how they compared. My hunch was that the zeros of $g(x)$ were six times greater than the zeros of $f(x)$, since that was the case with the vertices. **Table 5** summarizes my findings.

Now that my hunch had been confirmed, I excitedly began to look at other polynomials to

Table 5 Ratios of Zeros of $f(x)$ and $g(x)$

Function	$f(x)$	$g(x)$	Ratio $\frac{x_g}{x_f}$
Left Zero	$x_f = -\frac{4}{3}$	$x_g = -8$	$\frac{x_g}{x_f} = \frac{-8}{-4/3} = \frac{6}{1}$
Right Zero	$x_f = \frac{1}{2}$	$x_g = 3$	$\frac{x_g}{x_f} = \frac{3}{1/2} = \frac{6}{1}$

Table 6 Another Example: Comparison of Factors, Zeros, and Vertices

	Primary	Secondary
Trinomial	$3x^2 + 14x + 8$	$x^2 + 14x + 24$
Function	$f(x) = 3x^2 + 14x + 8$	$g(x) = x^2 + 14x + 24$
Factors	$(3x + 2)(x + 4)$	$(x + 2)(x + 12)$
Zeros	$x = -2/3, x = -4$	$x = -2, x = -12$
Vertex	$(-7/3, -25/3)$	$(-7, -25)$

Table 7 Zeros and Vertices of the Related Functions

Trinomial	$ax^2 + bx + c$	$x^2 + bx + ac$
Function	$f(x) = ax^2 + bx + c$	$g(x) = x^2 + bx + ac$
Zeros	$z_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ $z_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$	$z_3 = \frac{-b+\sqrt{b^2-4ac}}{2}$ $z_4 = \frac{-b-\sqrt{b^2-4ac}}{2}$
Vertex Abscissa	$x = -\frac{b}{2a}$	$x = -\frac{b}{2}$
Vertex Ordinate	$y = f\left(-\frac{b}{2a}\right)$ $= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$ $= \frac{-b^2+4ac}{4a}$	$y = g\left(-\frac{b}{2}\right)$ $= \left(-\frac{b}{2}\right)^2 + b\left(-\frac{b}{2}\right) + ac$ $= \frac{-b^2+4ac}{4}$
Vertex Coordinates	$V_f = \left(-\frac{b}{2a}, \frac{-b^2+4ac}{4a}\right)$	$V_g = \left(-\frac{b}{2}, \frac{-b^2+4ac}{4}\right)$

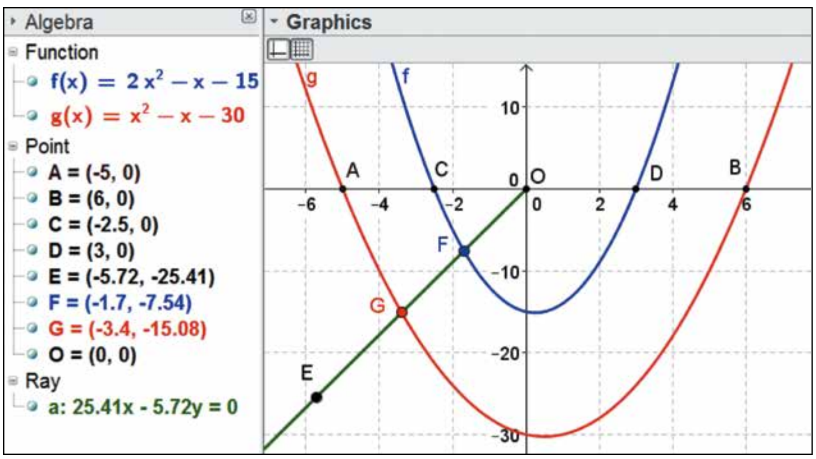

Fig. 2 Points F and G on ray OE have coordinates with a 2:1 ratio.

see whether the same relationships held. I went through the identical process with several other quadratic trinomials, one of which is shown in **table 6**.

A quick calculation showed that the ratio between corresponding zeros of the two functions was 3 : 1 and that the ratio of corresponding vertex coordinates was the same. This 3 : 1 ratio was also the ratio of the leading coefficients of the primary and secondary functions! For each additional trinomial that I investigated, the relationship between pairs of zeros and vertex coordinates of the two functions was the same. On the basis of these observations, I formulated the following hypothesis:

> The coordinates of the zeros and vertex of $g(x)$ are a constant multiple of the coordinates of the corresponding zeros and vertex of $f(x)$, and the value of the constant multiple is the same as the original leading coefficient of $f(x)$.

Armed with this working hypothesis, I went back and used a generalized analytical approach to verify the relationship for all such pairs of quadratic functions. **Table 7** shows the zeros of $f(x)$, labeled z_1 and z_2, and the zeros of $g(x)$, labeled z_3 and z_4. If you compare the larger zero of each pair (e.g., z_1 and z_3 when a is positive), you will see that they differ only by a factor of a. The same holds for the smaller zero of each pair (e.g., z_2 and z_4). This relationship leads us to the following conclusion about the zeros of the primary and secondary polynomial functions:

> The zeros of $g(x)$ are multiples of the zeros of $f(x)$ by a factor of a.

The vertex coordinates of the two functions are also related by a factor of a, the leading coefficient of the primary function. **Table 7** summarizes the process of finding vertices of each function.

The relationship between the pairs of zeros of the two functions is the cornerstone of the slip-slide method. We can use the multiple a to our advantage when we are factoring a quadratic whose leading coefficient is an integer greater than 1. Transforming quadratic functions rather than merely manipulating quadratic expressions allows us to factor quadratics graphically—this is *transformational factoring*.

TRANSFORMATIONAL FACTORING

Using polynomial functions and their zeros to produce corresponding binomial factors is at the heart of transformational factoring. The steps of transformational factoring may remind you of

Table 8 Factoring a Quadratic Trinomial by Transformation		
Step	$\boldsymbol{ax^2 + bx + c}$	$\boldsymbol{2x^2 - x - 15}$
1	Create the primary function $f(x) = ax^2 + bx + c$.	$f(x) = 2x^2 - x - 15$
2	Transform $f(x)$ into a secondary function, $g(x)$, where $g(x) = x^2 + bx + ac$.	$g(x) = x^2 - x - 30$
3	Factor $g(x)$.	$g(x) = (x - 6)(x + 5)$
4	Use the factors of $g(x)$ to find its zeros, z_3 and z_4.	$g(x)$ zeros: $z_3 = 6$; $z_4 = -5$
5	The zeros of $g(x)$ are multiples of the zeros of $f(x)$ by a factor of a; so divide the zeros of $g(x)$ by a to obtain the zeros of $f(x)$, z_1 and z_2. Simplify if possible.	$a = 2$ $f(x)$ zeros: $z_1 = 6/2 = 3$; $z_2 = -5/2$
6	Knowing the zeros of $f(x)$, create a pair of factors of the form $(x - z_1)(x - z_2)$.	$(x - 3)(x + 5/2)$
7	This product differs from the original trinomial by a factor of a, so multiply the product by a to obtain the factored form of $f(x)$.	$f(x) = 2 \cdot (x - 3)(x + 5/2)$ $f(x) = (x - 3)(2x + 5)$
8	The two binomials are the factors of the original trinomial, $ax^2 + bx + c$.	$(x - 3)(2x + 5)$

slip-slide factoring, but the method is rooted in graphical representations. The slip-slide method is an efficient routine that produces the desired factors, but the steps in the routine are not mathematically equivalent to one another. Unlike verifying a trigonometric identity, in which each step is equivalent to the preceding step, the slip-slide steps shown earlier in **table 2** are not equivalent. Thus, they do not constitute a proof that the product of final pair of binomials is equivalent to the original trinomial. Only when the functional approach is used does the method have meaning, leading us to factoring by transformation, the steps of which are shown in **table 8**. At the heart of transformational factoring is a dilation of one polynomial function to obtain a second related polynomial function.

The graphs of $f(x)$ and $g(x)$ shown in **figure 2** confirm the results of **table 8** and serve as a reminder of the relationship between the primary and secondary functions. Notice again the dilation transformation that occurs when the leading coefficient of the primary function, $f(x)$, is used to create the secondary function, $g(x)$.

This dilation transformation with respect to the origin has a scale factor equal to the value of the leading coefficient a. It can be shown algebraically that every point on the graph of $f(x)$ corresponds to a point on the graph of $g(x)$, with the distance from the origin to that point on $g(x)$ equal to a times the distance from the origin to the corresponding point on $f(x)$. **Figure 2** shows two such points, F and G, on $f(x)$ and $g(x)$, respectively. Notice how the coordinates of point G are twice the coordinates of point F. Both points lie on ray OE.

A NEW PERSPECTIVE

I frequently remind my student teachers that now is a great time to be a mathematics teacher. Tools that we now take for granted make it possible to investigate relationships visually, opening the door to new discoveries and new ways of looking at old ideas. The slip-slide method of factoring evolves into transformational factoring, a new and justifiable form of factoring graphically, thanks to our ability now to look at a long-standing problem from a new perspective.

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