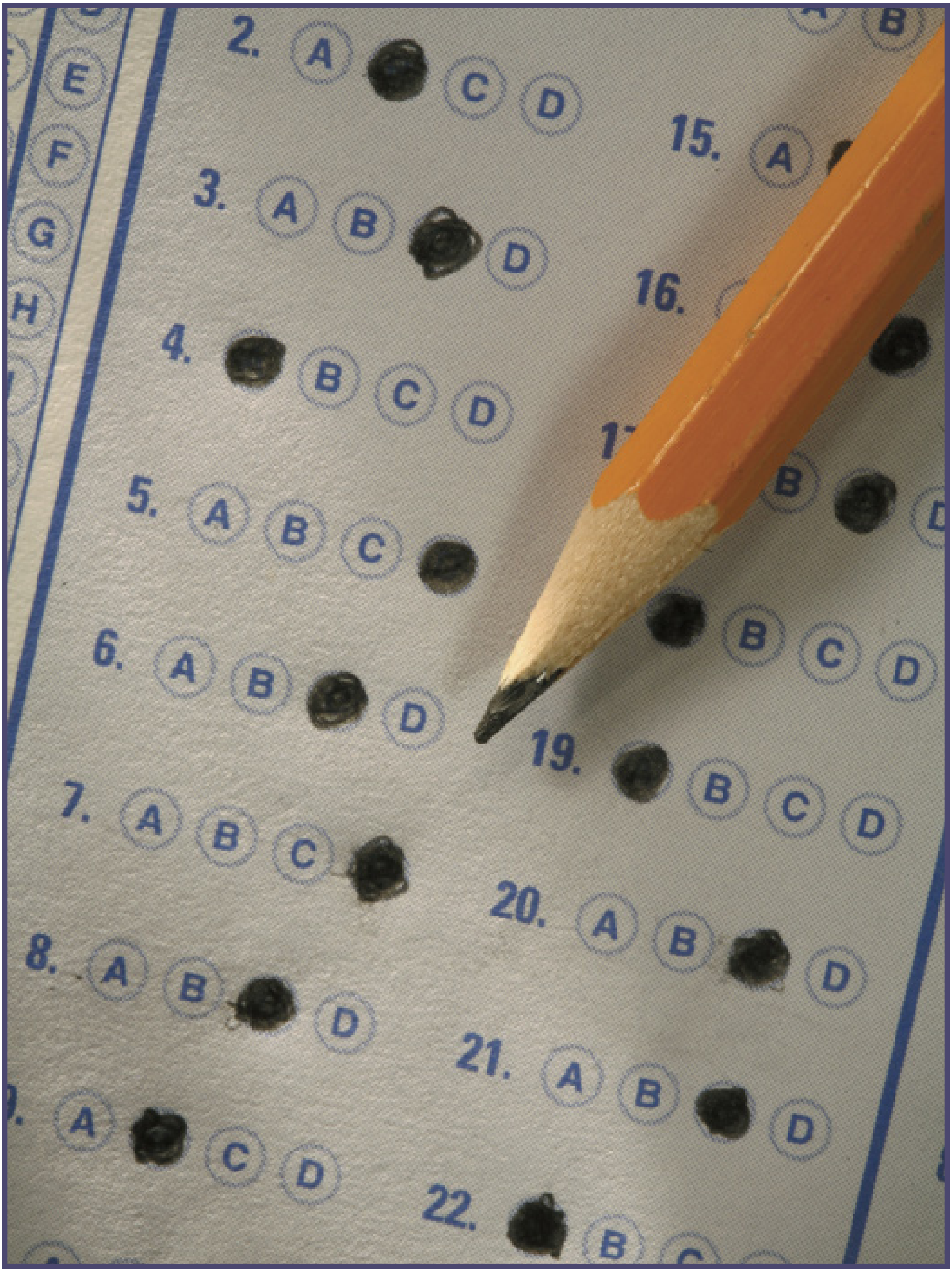

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Problems with *n*th-Term Problems

Examine sixth-grade students' work to gain insight into the early stages of their algebraic understanding.

Shirley M. Matteson

"What was it about *n*th-term problems that caused difficulty for my students?" After teaching a unit about *n*th terms to my sixth graders, I was frustrated to find that misconceptions continued to exist. I had hoped that exposing my students to various situations in which they generalized relationships between two sets of numbers would allow them to develop an understanding of variables, simple linear equations, and functions. Their difficulties left me asking myself if the best way to access students' thinking was to conduct individual interviews.

ANATOMY OF AN *N*TH-TERM PROBLEM

An *n*th-term problem involves a sequence (see **fig. 1**). Students must determine which expression will allow them to calculate the *n*th position of the sequence. To solve such problems, students are to find "a rule that determines the number of elements in a step from the step number" (Van de Walle 2007, p. 272). These types of problems help students develop concepts of functions, variables, and representations—some of the "big ideas" of algebra (Edwards 2000; Greenes and Findell 1999). Students also learn to generalize relationships between sets of numbers while developing an understanding of the various components of algebraic expressions, such as the variable and constant, by reading and writing algebraic expressions (see, for example, Lee and Freiman 2006; Smith, Hillen, and Catania 2007).

Goldin (2003) broadly defined *representation* as "a configuration of signs, characters, icons, or objects that can somehow stand for, or 'represent' something else" (p. 276). Van de Walle (2007, p. 276) enumerated five different representational types used in pattern or sequence problems:

1. The pattern itself, which we can refer to as the *context*;
2. The chart or table;

Reflect and Discuss

There Is an *n* in Algebra

Reflective teaching is a process of self-observation and self-evaluation. It means looking at classroom practice, thinking about what is done and why, and then evaluating whether it works. Collecting information about what goes on in the classroom and then analyzing and evaluating this information will allow teachers to identify and explore their practices and underlying beliefs.

The following questions are suggested prompts to help you, the reader, reflect on the article and on how the author's ideas might benefit your classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

This article models how teachers can work together to analyze student work, particularly when focusing on *n*th-term problems and their role in algebraic thinking.

- What sources for such problems and student work are available in your school?
- How can you use problems, such as those discussed in this article, with your students?
- How do you know when students understand that *concatenation* of a number and a symbol or two or more symbols implies multiplication?
- What balance is appropriate for teaching mathematics and teaching test-taking skills?

You are invited to tell us how you used "Reflect and Discuss" as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to *Mathematics Teaching in the Middle School* at **mtms@nctm.org**. Please include "Readers Write" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.

3. The symbolic equation;
4. The graph; and
5. The language.

Students may confuse or misinterpret the various mathematical representations used in presenting and solving *n*th-term problems. Common misunderstandings have included assigning inappropriate mathematical meaning to—

- *symbolic* representations in the form of $an + b$ or $3n$ (Chalouh and Herscovics 1999; Stacey 1989), in other words, misinterpreting the variable and any associated operations;
- *numerical* representations in the form of vertical or horizontal tables of numbers, such as treating each column or row as an independent number sequence;
- *graphical* representations of functions graphed on coordinate grids; and
- *verbal* representations, such as misconstruing verbal rules used to describe each pattern.

STRUCTURE OF THE RESEARCH STUDY

According to our state middle school math curriculum documents, students should use symbolic, numerical, graphical, and verbal representations to describe algebraic thinking (Texas Education Agency [TEA] 1998). The problem in **figure 1** appeared on our sixth-grade state math test in 2004 (TEA 2004). Of the almost 290,000 students in the state who answered this problem, only 48 percent selected the correct answer.

Students in other grade levels also had difficulty solving *n*th-term problems. The problem in **figure 2** is essentially the same (TEA 2003) but contains a change in table orientation and uses verbal rules instead of algebraic

Fig. 1 Only 48 percent of 290,000 students correctly answered this sixth-grade test item.

34 What is the rule to find the value of a term in the sequence below?

Sequence

Position, *n*	Value of Term
1	1
2	4
3	7
4	10
5	13
n	?

F $n + 3$
G $3n - 2$
H $3n$
J $n - 2$

Released Test Item—Grade 6, 2004 TAKS Test. Copyright © Texas Education Agency. All rights reserved.

Fig. 2 About 49 percent of 280,000 students correctly answered this seventh-grade test item, which is oriented differently and worded differently than the question in **figure 1**.

35 Which description shows the relationship between a term and *n*, its position in the sequence?

Position	1	2	3	4	5	*n*
Value of term	1	4	7	10	13	

A Add 3 to *n*
B Multiply *n* by 2 and then subtract 3
C Multiply *n* by 2 and then add 3
D Multiply *n* by 3 and then subtract 2

Released Test Item—Grade 7, 2003 TAKS Test. Copyright © Texas Education Agency. All rights reserved.

expressions for the answers. This item appeared in 2003 on a seventh-grade state math exam. Only 49 percent of over 280,000 students correctly solved this version of the problem. These phenomena became a catalyst for exploring the representational features of standardized assessment problems. The investigation presented here reflects a portion of a study undertaken during my doctoral program in math education, in which university professors interviewed sixth-grade students about their understanding of the problem in **figure 1**.

Two math education professors interviewed a stratified random sample of twenty sixth-grade middle school students. I was one of three sixth-grade math teachers at the school, and a few of the twenty were students of mine. The purpose of the interview was to access students' understandings and interpretations of the mathematical representations in several standardized assessment problems and find out how students used this information to select their answers. The nth-term problem was one of four that students solved before the interview. The other three problems used other mathematical representations, such as diagrams and geometric shapes.

Students were encouraged to think aloud about their solutions, and the interviewers focused on clarifying what students were saying about the representations. The four problems had appeared on a midyear benchmark test administered two months before the interview to give math teachers an overview of how well students understood specific math concepts. The student work on those items was also analyzed. We three math teachers continued to address mathematical concepts and representations between the administration of the benchmark test and the interview.

Of the twenty students interviewed about the problem in **figure 1**, eight either wrote or verbalized a pattern of "+ 3" between the numbers in the right-hand column and chose F. Seven students selected the correct answer of G. These students either wrote or verbalized that one would multiply the number in the left column by 3 and then subtract 2 to get the number in the right column. Five students found a pattern between the two set of numbers of +0, +2, +4, +6, and +8. Of this group, two students chose F, one chose H, and the other two did not make a selection because none of the answers matched their patterns.

In the following section, excerpts of interviews and written work from six of the twenty students provided insights into students' thinking. All student names are pseudonyms.

Amber's written work presented a unique interpretation of the symbolic notation in the nth-term problem. Sharon exemplified correct interpretations of representations. Angelina's verbal approach to identifying relationships within a given sequence provided a forum for discussing instructional and testing-taking strategies. The work of Sylvia, Marcus, and Sam illustrated the variety of students' thinking about the patterns in the problem.

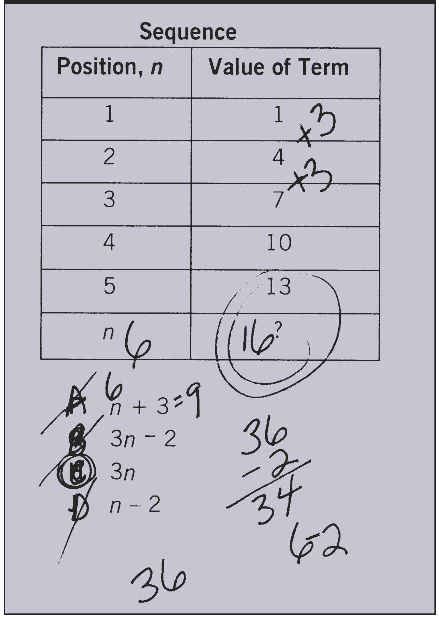
EXAMPLES FROM THE STUDY

Symbolic Notation Issues

Although I incorporated a discussion on place value when teaching students about substituting numbers for a variable, some students applied place-value representations in an algebraic context. For example, students regarded the "n" in nth-term problems as an abbreviation for the word *number*. This concept was reinforced in students' minds when learning that in the formula $A = lw$, we multiply l by w instead of incorrectly seeing lw as a two-digit number.

Amber's written work demonstrated this misconception (see **fig. 3**). She interpreted n as being 6 by extending the left-hand side of the table. As with many students, she assumed that n was place-value specific and could only appear in the ones place and that substituting a number for the n would result in a specific number. For choice A, she wrote 6 + 3 = 9, choice B was 36 – 2 = 34, C was 36, and D was 6 – 2. Amber had substituted the 6 both times for $3n$ in each expression as 36. In her mind, each iteration of the $3n$ expression would have been a series of two-digit numbers starting with 31, 32, 33, 34, 35, and 36.

Fig. 3 Amber's solution to question 34 involved the misconception that n represented a single digit and was place-value specific.



A further examination of Amber's work revealed that she had assumed a final right-column answer of 16 for $n = 6$, as shown by the +3 by the first few values in that column. She actually crossed out choice C but had to select one answer after eliminating all of them. During the interview, Amber clarified that she was confused by the symbolic notation and had simply selected a final answer even though, in her mind, all answer choices were wrong. The interviewer worked with

Amber to help her understand the correct meaning of the $3n$ notation. When she understood the concept, she chose the correct answer.

Students can frequently manipulate numbers to solve problems using formulas such as $A = lw$ but may not be able to understand the meaning of symbol combinations such as $3n$ or xy in other situations. More than half the interviewed students were unable to explain the meaning of $3n - 2$ and $3n$ that appeared in the answer choices.

Students are generally exposed to this type of notation at the elementary level. For example, our state's fourth-grade math test asked students to use the perimeter formula for a rectangle of $P = 2l + 2w$. When symbolic representations are inadequately explained or ignored, students such as Amber may develop faulty understanding. They should be able to explain representations in their own words, not just respond with a textbook definition. In contrast, Sharon's explanation demonstrated her knowledge of the "missing" operation as multiplication.

Interviewer: Where did you learn that the three next to a letter meant to multiply?

Sharon: In math. It doesn't necessarily have to have a multiplication sign or a dot or something that separates it. It just has to have a number and either two numbers or a number and a letter or two letters. You've just gotta know what the letter means. You have to be specific in that.

Sharon demonstrated her understanding of the symbolic notation including combinations of letters and/or numbers, several ways of representing the multiplication sign, and the importance of knowing the meaning of the variable to find the solution. This critical step is often overlooked by students when solving problems. My desire is to have all students reside at this level of representational understanding.

One informal assessment strategy I have used to access my students' understanding of symbolic representations, such as $3n$ or xy, has been to write $A = lw$ on the board. Students explain, "That's the formula for finding the area of a rectangle. You know, area equals length times width." However, they are often puzzled by my follow-up question: "Where's the times sign?" Students' explanations have given me critical information concerning their knowledge of symbolic notation.

Although students have worked with several formulas in previous grade levels, the idea of a variable referring to any unspecified number appears to have been underemphasized. Students often substitute specific numbers in the length and the width, multiply, and determine the solution. However, they often have difficulty if they do not have a length or a width or are not given a number for the area so as to work backward to find one or more of the dimensions.

Some students have not been formally introduced to *concatenation*, the concept that a letter next to a letter, for example, xy, or a number next to a letter, $3n$, means to multiply. Therefore, it is important to assign tasks that allow students to become confident with creating or interpreting algebraic expressions. One such task I have used is to give students a list of types, sizes, and prices of movie theater snacks and ask them to algebraically represent buying various combinations.

Relationships

The format of the nth-term problem in **figure 1** did not require students to create a rule on their own, which many of the interviewed students ignored. Students needed only to select a rule from the solutions that were provided. However, most students worked to find a pattern but then struggled to generalize a rule to explain the relationship between the right-hand and left-hand columns. Sylvia's pattern (see **fig. 4a**) of adding 3 is a typical response that leads

Fig. 4 By examining the sketches that students made on their tables while working on question 34 (see **fig. 1**), we can learn a lot about how they think when solving nth-term problems.

Sequence

Position, n	Value of Term
1	1+ 3
2	4 +3
3	7 +3
4	10 +3
5	13 +3
n	? 16

(a) Sylvia

Sequence

Position, n	Value of Term
1	1 0
2	4 2
3	7 4
4	10 6
5	13 8
6 n	16 ? 10

(b) Marcus

Sequence

Position, n	Value of Term
1 ×3=	3-2 = 1
2×3=6	-2 = 4
3×3=9	-2 = 7
4×3=12	-2 = 10
5×3 -15	-2 = 13
6 n ×3=	18-2= ?

(c) Sam

to choosing $n + 3$ as the solution. Unfortunately, this pattern ignores any relationship between the first and second column of numbers.

Marcus's pattern (see **fig. 4b**) of 0, 2, 4, 6, 8, and 10 resulted from adding successive even numbers to the position number to match the value-of-term number (which is also the difference between the values in each row). Students with similar solutions had difficulty selecting an answer because none of the choices included "+ 2." Sam's work (see **fig. 4c**) showed that he multiplied n by 3, then subtracted 2. Although this was a correct solution, he demonstrated an inappropriate use of the equal sign in a series.

Since pattern problems may be described by more than one expression, students needed a strategy for checking possible solutions that were already provided on standardized assessments, especially if they derive an expression not listed among the choices. Angelina's approach was to work backward and evaluate the accuracy of each expression to determine the correct answer.

Interviewer: Tell me about this problem.

Angelina: I usually don't look at my answer choices, but I had to look at the answer choices 'cause I was kind of stuck. And then I went through every one and see if it fit.

Interviewer: Give me an example of what you meant.

Angelina: I usually try to figure it out by myself. I thought it was 1×1 and then like going down 2×2, but then when I got to 3×3 it was 9 not 7.

Interviewer: So how did you use your answer choices?

Angelina: I knew this was 6 so I crossed it out and made it a 6. And then $n + 3$ would be lower, it would be 9, I think, yeah. It'd be 9. And it would be lower. And it couldn't be lower 'cause it's going higher, so I knew that wasn't right. And then I had skipped over G. And I kept going down.

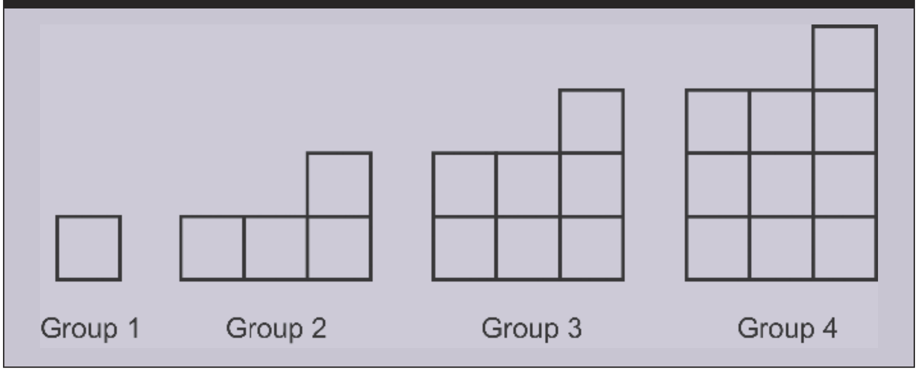
Other students who worked through the solution choices also successfully answered this problem, so I decided to teach Angelina's strategy to all my students. Although readers may dismiss her method as simply a test-taking strategy, it is effective when incorporated within classroom discussions. I asked students questions about the meaning of the various components of the expressions, including the variable, the operators, and the constant. I wanted to ensure that everyone understood the terminology and could provide explanations and examples in their own words. Students also started applying the strategy to other types of challenging problems. I observed students making fewer careless mistakes, looking at all multiple-choice answers even if they thought they knew the solution, and marking reasons for rejecting answers. Student performance on final assessments on *n*th-term problems improved, and I was more satisfied with my teaching efforts.

CLOSING REMARKS

Students in the middle grades must become adept at understanding and interpreting various mathematical representations. Engaging students in discussions that probe their understandings of representations may be time-consuming but is beneficial for both teachers and students. Such *n*th-term problems give students opportunities to interpret symbolic representations and generalize patterns and relationships, which exposes them to the foundations of algebra. Early success with algebra creates a positive experience and encourages students to persevere in math. "Algebraic competence is important in adult life, both on the job and as preparation for postsecondary education" (NCTM 2000, p. 37).

Math teachers must create positive initial experiences with algebra and should be prepared to help students understand important concepts, such as how $3n$ is related to $A = lw$ or how individual patterns become related to functions. Teachers must also incorporate activities that require students to gain fluency in using and understanding many different representational forms. As demonstrated in **figures 1** and **2**, tables and rules are the building blocks of graphical relationships, which are usually derived from a physical model (Lee and Freiman 2006; Smith, Hillen, and Catania 2007), such as the set of blocks that appear in **figure 5**. After becoming familiar with these representations, the results from the table can become a set of ordered pairs that can be graphed on a coordinate plane. Math teachers should also include opportunities for students to

Fig. 5 This geometric representation of sixth-grade test item 34 models the problem.



use these representations in *n*th-term activities (Rubenstein 2002).

The problem featured in this study contained multiple representations and was from a standardized assessment. However, students need experience with multivariable real-world problems. Blanton and Kaput (2004) found that "the typical emphasis on pattern finding in single variable data sets in early elementary grades might impede an emphasis on functional thinking in later elementary grades and beyond" (p. 141). Students need to explore patterns with objects and situations that have a variety of features and investigate the relationships between multiple sets of numbers.

This study focused specifically on students' understandings of representations in standardized assessment problems. Math educators should engage students in discussions about similar types of problems. If state tests are not available, teachers can examine problems from the Trends in International Mathematics and Science Study (TIMSS) and National Assessment of Educational Progress (NAEP) found on the Web site of the National Center for Education Statistics (NCES). However, the purpose of examining assessment items should be to ascertain students' understandings of representations, not teach to a test.

Discussions can also focus on the appropriateness of the representation, other means of representing the information, or how to represent the problem or solution to a younger or older person. Engaging in meaningful mathematical discussions is time-consuming, and middle-level students may need to be encouraged to verbalize their thinking. However, discussions about mathematical thinking will benefit both students and teachers.

You and your students may find a greater enjoyment in exploring mathematical concepts, such as those imbedded in *n*th-term problems.

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