THE MATHEMATICS OF POKER: BASIC EQUITY CALCULATIONS AND ESTIMATES

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INTRODUCTION

The game of poker has experienced a worldwide boom in popularity in the past decade. This can be attributed to a confluence of events: the invention of the "lipstick" camera that allows viewing of players' "hole" (hidden) cards for television broadcasts, the growth of online poker rooms, and the increased popularity of "satellite" events that allow players to win their way into larger-money events. The largest single prize awarded in a poker tournament to date, 12 million US dollars, was won by an amateur player in 2006 [2].

Poker is a game of skill that contains an element of chance; it lies somewhere between chess (pure skill) and roulette (pure luck) in the spectrum of games. Since poker is a game of incomplete information, skilled players often use psychology to help them make correct decisions or to influence others to make incorrect decisions. Some situations do arise, however, in which mathematical considerations are paramount. It is the purpose of this article to discuss such situations. Since the first section covers the rules of poker in detail, the subsequent discussion should be of interest and accessible to students with only a basic knowledge of combinatorics and probability.

THE RULES

Modern poker is played with a standard deck of 52 cards, using cards marked with numbers 2 through 10, J (Jack), Q (Queen), K (King), and A (Ace) to indicate, in increasing order, their respective ranks. Aces may be ranked below 2s if helpful to the player. There are 4 cards of each rank, bearing each of the 4 suits: ♣ (clubs), ♦ (diamonds), ♥ (hearts), and ♠ (spades). Although there are many variations of the game of poker [3], the most common ones use the best 5-card combination ("hand") for each player to determine the winner, with the hand rankings given in decreasing order as follows:

Royal flush: a hand containing the 5-card sequence 10, J, Q, K, A of the same suit

Example: 10♠, J♠, Q♠, K♠, A♠.

Straight flush: a hand containing 5 cards in sequence and of the same suit, but not a royal flush

Example: 2♥, 3♥, 4♥, 5♥, 6♥.

Four of a kind: a hand containing 4 cards of the same rank and one card of another rank

Example: J♠, J♥, J♦, J♣, A♣.

Full house: a hand containing 3 cards of one rank and 2 cards of another rank

Example: 7♣, 7♦, 7♥, 2♥, 2♦.

Flush: a hand containing 5 cards of the same suit that are not in sequence

Example: 2♣, 3♣, 4♣, K♣, A♣.

Straight: a hand containing 5 cards of sequential rank, not all of the same suit

Example: A♣, 2♥, 3♠, 4♣, 5♣.

Three of a kind: a hand containing 3 cards of one rank and two cards of two other different ranks

Example: K♦, K♥, K♠, A♠, Q♦.

Two pair: a hand containing 2 cards of one rank, 2 cards of another rank, and a fifth card of a third rank

Example: 9♦, 9♣, 8♦, 8♣, 7♣.

One pair: a hand containing 2 cards of one rank and 3 cards of 3 other different ranks

Example: 5♣, 5♠, A♣, K♥, J♣.

High card: a hand containing none of the preceding hand combinations

Example: 2♣, 4♥, 7♠, K♠, A♣.

Within hands of the same category, hands are ranked using card rankings. For example, the hand K♦, K♥, K♠, A♠, Q♦ has a higher rank ("beats") the hand J♦, J♥, J♠, 10♠, 9♦ because Kings are ranked higher than Jacks. Similarly 9♣, 10♥, J♠, Q♣, K♣ beats A♣, 2♥, 3♠, 4♣, 5♣. Since cards can be shared, it can occur that one player has K♦, K♥, K♠, A♠, Q♦ while another has K♦, K♥, K♠, 10♠, 9♦. In this case, the first player wins because the Ace "kicker" outranks the 10 kicker. Likewise, the hand 5♣, 5♠, A♣, K♥, J♣ beats 5♣, 5♠, A♣, Q♥, J♣ because the King outranks the Queen. Since players can have the same hand, ties are possible.

While the categories of hand rankings may be somewhat arbitrary (why not have one for even numbers or red cards only, for example?), the rankings themselves are not. The order of the categories in the preceding list is based on the probabilities of each of the hands occurring. For instance, there are exactly 4 ways (one for each suit) in which a player can obtain a royal flush out of a total of ${}_{52}C_5 = 2598960$ possible 5-card hands, making the probability of a royal flush $4/2598960 \approx 1.54 \text{ x } 10^{-6}$. Likewise, the probability of a straight flush is $36/2598960 \approx 1.39 \text{ x } 10^{-5}$ (why?). It is an interesting exercise to determine the probabilities of being dealt a hand in each of the other categories.

Currently, the most popular variant of poker is Texas Hold 'Em. In this game, each player is dealt 2 "hole" cards face down, and can then act on his/her cards after looking at them. Almost always, players are required to place "blind" bets before seeing their cards according to some well-defined system so that players do not simply wait for the best starting hands before playing. Other players may choose to increase the stakes to play ("raise"), continue playing at the same stakes ("call"), or decline to continue playing ("fold"). After the first betting round, 3 "community" cards (the "flop") are dealt face up, to be used with each player's hole cards to make a 5-card hand. After another round of betting, another community card (the "turn") is dealt face up, so that new combinations of best 5-card hands are possible. There is a third betting round, a fifth community card (the "river") dealt face up, and a final betting round. If at any point only one player remains due to all other players folding, that player is awarded all the bets made in that round (the "pot"). Otherwise, the best hand(s) is (are) awarded the pot in the "showdown", when players show their hole cards after the last betting round.

The betting structure itself also varies. "Limit" versions of Texas Hold 'Em use a fixed betting structure. Usually bets and raises consist of one unit (dollar, euro, M&M, etc.) in the first 2 betting rounds, and 2 units in the last 2 betting rounds. "No-limit" versions allow players to bet everything they have. The different betting structures can strongly influence the mathematically correct decision in certain situations.

Case Study I: a flopped straight versus an ace-high flush draw

For our first example, suppose that there are only two players. Player A holds hole cards 5♦, 6♦ while Player B holds K♣, A♣, and the flop is 4♣, 7♣, 8♠. So after the flop, Player A's hand is a straight (4♣, 5♦, 6♦, 7♣, 8♠) while Player B only has a high card Ace (4♣, 7♣, 8♠, K♣, A♣). Now suppose that there is a no-limit betting structure and that Player A bets all his/her remaining units, exposing the 5♦, 6♦ and advising Player B to fold. Should Player B comply? The answer is simple: *It depends.*

Notice that Player B only needs one more club to make a flush (which beats a straight), and that there would be two more cards to come should he/she call Player A's "all-in" bet. If the probability that Player B will win the hand and the amount in the pot are both large enough relative to the amount required to call, then Player B should call (or "draw to" the flush). For example, suppose that the pre-flop betting was such that 10 units were in the pot, and that Player A's all-in bet was another 10 units. Thus, Player B has to call 10 units to win the 20 units in the pot (often, poker players would say that Player B is getting 20-to-10 or 2-to-1 *pot odds* to call in this situation). So if Player B will win at least

$$\frac{\text{amount to call}}{\text{amount in the pot after the call}} = \frac{10}{10+20} = \frac{1}{3} \approx 33.3\%$$

of the pots on average (that is, if Player B's *equity* is at least 33.3 %), a call is in order. Let's do the calculation in this case, and discuss the results. There are three mutually exclusive and exhaustive possibilities:

1) The players tie: Although the chances of this are very small, the probability is still nonzero. The players will tie if the turn and river cards consist of a 5 and a 6, neither of which is a club. There are 4 ways that this can happen out of a total of ${}_{45}C_2$ possible two-card combinations from the 45 remaining cards in the deck. The players tie with a probability of $\frac{4}{{}_{45}C_2} = \frac{4}{990} \approx 0.004$.

2) Only Player B wins: This occurs if a club is dealt either on the turn or the river (or both). Since there are 9 clubs remaining, the probability that a club will be dealt is

$$1 - P(\text{no club is dealt on the turn or river}) = 1 - \left(\frac{36}{45}\right)\left(\frac{35}{44}\right) \approx 0.364.$$

Here, we use the fact that 36 non-clubs can be dealt on the turn, and 35 non-clubs remain to be dealt on the river after a non-club is dealt on the turn.

3) Only Player A wins: This event is the complement of the previous two events, so its probability is

$$1 - P(\text{the players tie}) - P(\text{only Player B wins}) \approx 1 - 0.004 - 0.364 = 0.632.$$

Since Player B will win the whole pot approximately 36.4% of the time if he/she calls, calling is the mathematically correct action. Players often look at this via pot odds: is Player B getting better than 2-to-1 odds to call? Here, however, "better than" means "less than". Neglecting the small probability of a tie, the odds of Player B winning are 0.632-to-0.364, or approximately 1.74-to-1. The odds favor a call.

Taking these calculations one step further, we can compute Player B's expected earnings in units in the case of a call by Player B:

$$E(\text{Player B winnings}) \approx 0.364(20) + (0.632)(-10) + 0.004(15) = 1.02.$$

In other words, a call by Player B results in a gain of 1.02 units for Player B on average.

If the situation is changed slightly, so that there are 10 units in the pot pre-flop, but Player A bets 20 units all-in, then the pot odds for Player B's call change to 30-to-20, or 1.5-to-1. In this case, calling is a losing proposition in the long run for Player B. In fact, now we find Player B's expected earnings to be

$$E(\text{Player B winnings}) \approx 0.364(30) + (0.632)(-20) + 0.004(25) = -1.62.$$

This example also illustrates the major difference between limit and no-limit betting structures. The latter structure allows players to adjust the pot odds more easily to make an opponent's call a bad (and more expensive) decision.

Case Study II: a flopped flush versus three of a kind

Suppose now that Player A holds K♠, A♠, while Player B holds 10♣, 10♦ in a two-player round. If the flop is 2♠, 3♠, 10♠, both players hold fairly strong hands: Player A has an ace-high flush, while Player B has three of a kind. Although Player A is currently ahead, Player B can still win the pot by making either a full house or four of a kind (i.e., "the board pairs") by the showdown. Assuming that the hand is played to showdown, what is Player B's equity in this case? Since there cannot be a tie, Player B's equity is the same as Player B's probability of winning the pot. To calculate the probability of this event E we decompose it into three disjoint sub-events:

E_1: the event that the board pairs on the turn,

E_2: the event that the board does not pair on the turn, but pairs on the river with a card that is neither an ace nor a king, and

E_3: the event that the board does not pair on the turn, but pairs on the river with an ace or a king.

There are 7 cards that can pair the board on the turn. If the board does not pair on the turn, there are 32 cards that are not an ace or a king that do this, and 6 aces/kings that do this. If a card that is not an ace or a king comes on the turn, then the board can pair on the river with 10 cards; if an ace or king comes on the turn, then the board can pair on the river with 9 cards. Thus, we calculate:

$$P(E) = P(E_1) + P(E_2) + P(E_3) = \frac{7}{45} + \left(\frac{32}{45}\right)\left(\frac{10}{44}\right) + \left(\frac{6}{45}\right)\left(\frac{9}{44}\right) \approx 0.344;$$

i.e., Player B has about 34.4 % equity in this situation.

PRACTICAL CONSIDERATIONS: THE RULE OF FOUR-TWO EXPLAINED

While nontrivial in nature, the case studies discussed here only begin to illustrate the complexity possible in poker. For example, absolute knowledge of another player's hand is very rare; usually, one can only make an intelligent guess about an opponent's hole cards based on previous play. A player can "bluff" by making a large bet to falsely indicate to others that his/her holdings are very strong, inducing others to fold better hands. The resulting "fold equity" is very real, but difficult to calculate. Also, the possibility of multi-player pots can further complicate analysis. Finally, players do not have the luxury of time (or pen, paper, calculator, computer, etc.) to calculate equity or odds precisely as we have done above. Some players memorize equity tables for commonly occurring situations. Fortunately, there is a well-known way to estimate equity by counting "outs", which are cards that will likely improve a player's currently losing hand to a winning hand.

The *rule of four-two* [1] provides a simple way to estimate one's probability of winning at showdown given that m outs are available among the unseen cards. The rule is applied as follows: given m outs after the flop, the probability of winning if the hand is played to showdown is approximately $4m$ percent, or $0.04m$; given m outs after the turn, the probability of winning if the hand is played to showdown is approximately $2m$ percent, or $0.02m$. For example, Player B in Case Study I above has 9 outs after the flop. Applying the rule of four-two, Player B's equity in the hand is approximately $(4)9\%$, or 36%. If a card such as 2♦ comes on the turn, then the rule of four-two estimates the probability of Player B winning the hand at showdown to be 18% (Exercise: How does this compare with the actual answer?).

Why does this rule work? If m outs are available to a player after the flop, the event F of that player "hitting" an out on either the turn or the river can be decomposed into two disjoint sub-events:

F_1: the event that the player hits an out on the turn, and

F_2: the event that the player does not hit an out on the turn, but hits an out on the river.

Because there are 47 unseen cards (i.e., we do not assume absolute knowledge of any opponent's hole cards), the probability of event F, which approximates the probability the player wins the hand in a showdown, is

$$P(F) = P(F_1) + P(F_2) = \frac{m}{47} + \left(\frac{47-m}{47}\right)\left(\frac{m}{46}\right) = \frac{93m - m^2}{(47)(46)}.$$

For typical values of m, m^2 is small relative to $93m$. Thus, one can use the approximation

$$P(F) \approx \frac{93m}{(47)(46)} \approx 0.04m.$$

The following table compares the approximation with the actual value of $P(F)$.

m	**0.04m**	$P(F)$	**% error**
1	0.04	0.042553	-6
2	0.08	0.084181	-4.96703
3	0.12	0.124884	-3.91111
4	0.16	0.164662	-2.83146
5	0.2	0.203515	-1.72727
6	0.24	0.241443	-0.5977
7	0.28	0.278446	0.55814
8	0.32	0.314524	1.741176
9	0.36	0.349676	2.952381
10	0.4	0.383904	4.192771
11	0.44	0.417206	5.463415
12	0.48	0.449584	6.765432

A similar analysis can be done for the "two" part of the rule of four-two.

In Case Study II above, the number of outs actually increases when the flop card is dealt (from 7 to 10 if no out is hit). Thus, it is reasonable to take the average number of outs, 8.5, in the calculation after the flop. This yields a result of 4(8.5)% = 34%, which compares quite favorably with the exact equity value.

CONCLUDING REMARKS

We have considered ways to calculate exactly and to estimate a poker player's equity in certain two-player Texas Hold 'Em situations. Many other distinct situations are worthy of analysis. One possibility is "top pair, top kicker versus an overpair" as in the following example: Player A holds 10♠, A♠, Player B holds Q♦, Q♣, and the flop comes 10♣, 7♦, 2♥. Other commonly occurring two-player situations may be studied, and the exact equity calculations compared with the rule of four-two estimates. Alternatively, three-player cases may be analyzed. Finally, the rule of four-two may be modified to yield better equity estimates without significantly more involved calculations.

REFERENCES

1. Bill Burton, *From the Kitchen to the Cardroom- Get the Edge at Low Limit Texas Hold'em,* Bonus Books Inc., ISBN: 1566251893 (2003).
2. Peter Sharkey, "Jamie Strikes Gold with Biggest Prize; Play your Cards Right", *USA Today*, p. C10, ISSN: 07347456 (September 26, 2006).
3. Stewart Wolpin, *The Rules of Neighborhood Poker According to Hoyle*, New Chapter Press, ISBN: 0942257197 (1990).