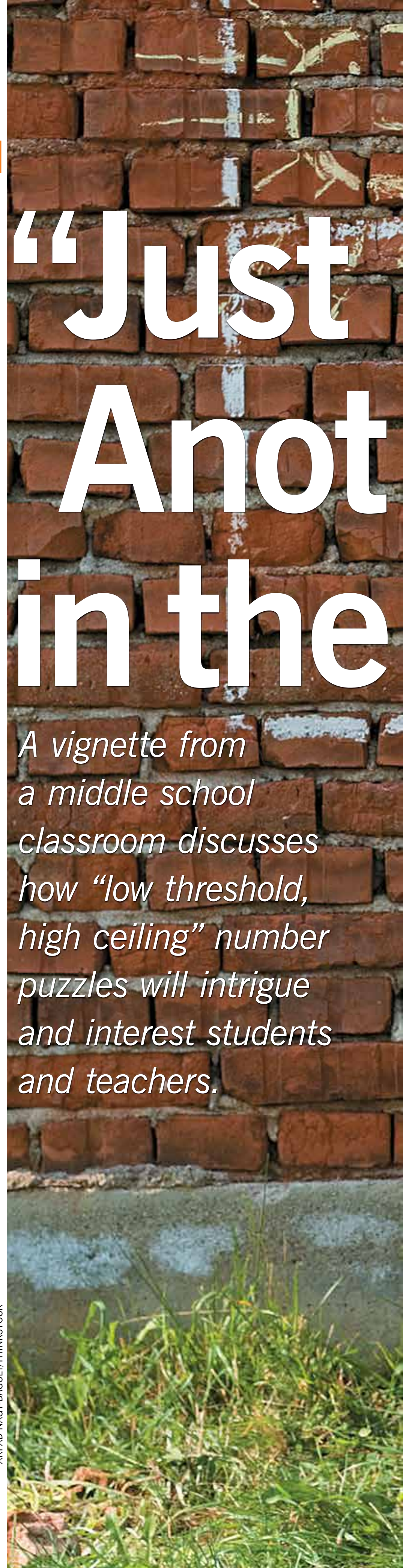
NOT “Just Anot in the



A vignette from a middle school classroom discusses how “low threshold, high ceiling” number puzzles will intrigue and interest students and teachers.

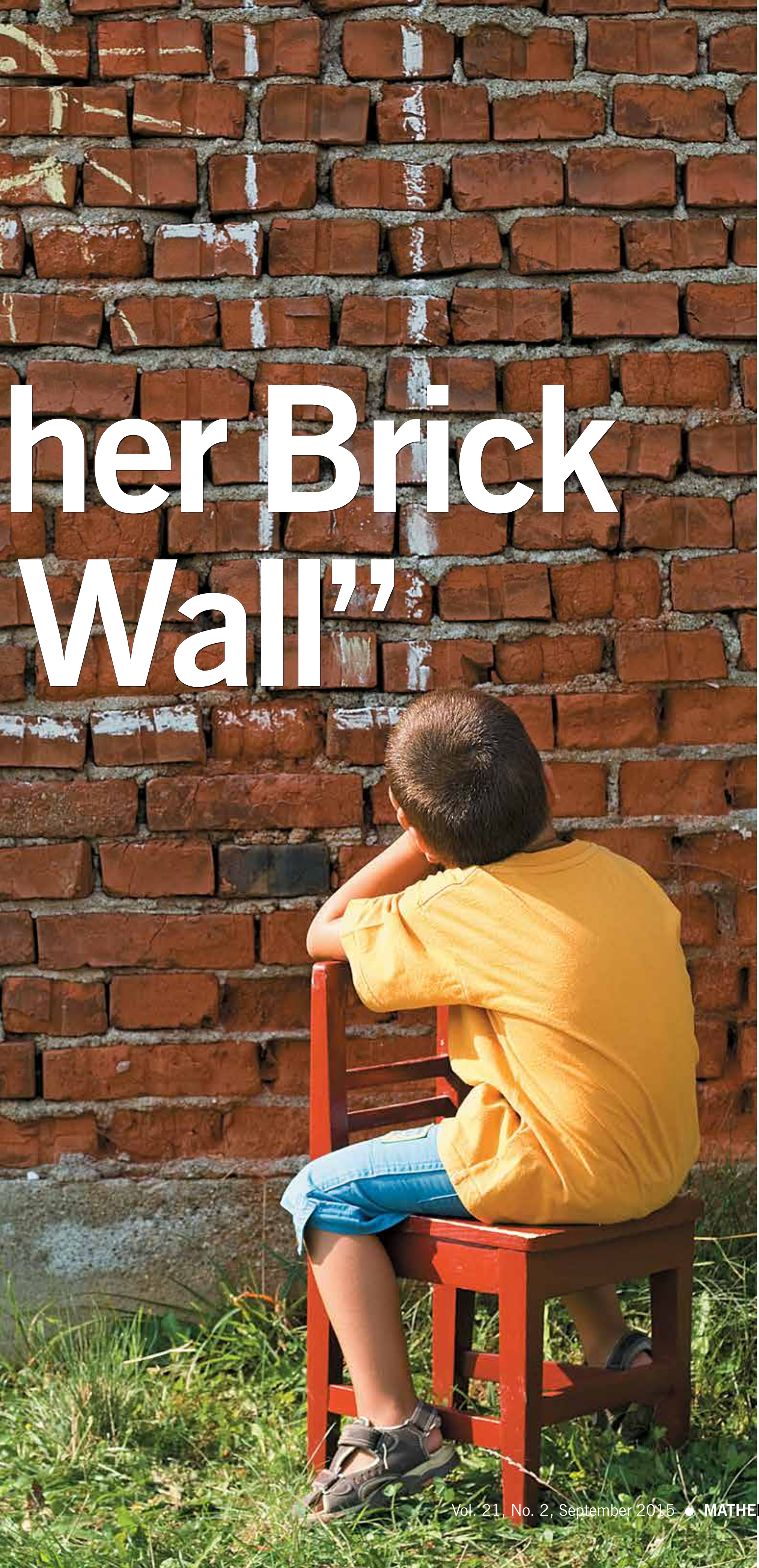
Betina A. Zolkower and Laurie H. Rubel

ARPAD NAGY-BAGOLY/THINKSTOCK

“Low threshold, high ceiling” tasks (McClure 2011) are accessible to diverse learners; invite a wide range of approaches; and hold the potential to further challenge, strengthen, and extend everyone’s mathematical reasoning. We present a family of Brick Pyramid problems (Wittmann 1995; Selter 1997; Müller 2003) as examples of “low threshold, high ceiling” tasks. By eliciting the practices of noticing and describing patterns and then symbolizing and generalizing those patterns, Brick Pyramid problems hold great potential for engaging students in “algebraizing” (Freudenthal 1991).

The pyramid in **figure 1** has five bricks in the bottom row, and each subsequent higher row contains one fewer brick. Numbers are assigned to bricks according to the following rule

her Brick Wall”



(see **fig. 2**): The number on each brick from the second up to the top row is the sum of the numbers from the two bricks directly beneath it.

A completed pyramid is shown in **figure 3**. By arranging 1, 2, 3, 4, and 5 on the bottom row in ascending order, the top brick results in 48. If we rearrange 1, 2, 3, 4, and 5 on the bottom row, would the top brick remain 48? Before reading further, we encourage readers to investigate different arrangements of 1, 2, 3, 4, 5 in the bottom row. Which arrangement results in the maximum number for the top brick? The template provided in **figure 1** can serve as a tool for keeping track of your work. Look for patterns and try to make sense of the patterns you notice.

In the following section, we present a vignette that describes, from the perspective of the facilitator (author Zolkower), an experience using the brick pyramid task in a New York City eighth-grade public school classroom.

A LESSON VIGNETTE

I began by drawing on the blackboard a five-level pyramid and then writing the numbers 1, 2, 3, 4, and 5 in ascending order, one on each of the five bricks in the bottom row. Next, I entered the sums to complete the pyramid, as shown in **figure 3**.

I turned to the class and asked, “What am I doing here?” Students seemed intrigued, perhaps because the brick structure was something new or because it was easy to make sense of the rule for filling it in. Christina volunteered, “You add up the two numbers below each brick.” Others nodded in agreement.

I then drew a second brick pyramid with a different ordering of the bottom row numbers (1, 3, 4, 2, and 5) and asked the class, “What do you think is the problem I want you to solve?” Nelson replied, “You want us

The additive structure of the brick pyramid drew the students in. The fact that a question had not yet been posed added to the intrigue.

Fig. 1 A brick pyramid awaits the addition of numbers.

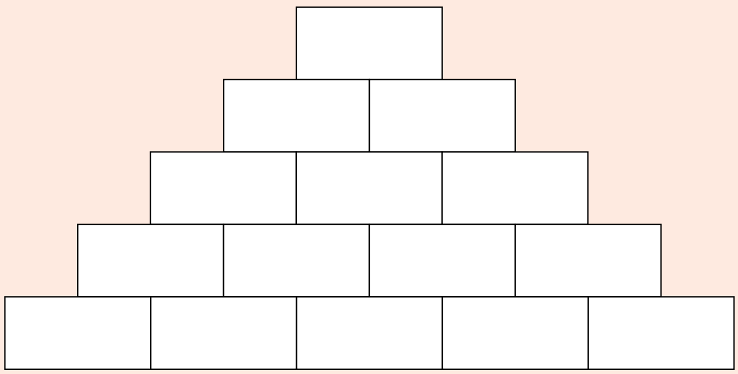


Fig. 2 The addition rule for the brick pyramid is illustrated.

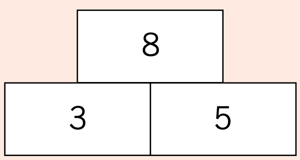
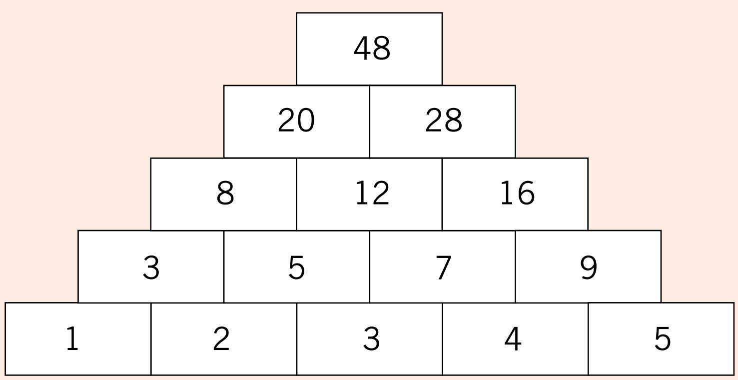


Fig. 3 A 1, 2, 3, 4, 5 arrangement was given to students.



to fill in the bricks to get different answers." Charles commented, "That would be too easy. . . . That's just adding numbers!" "Exactly," I agreed, "you are in eighth grade, and it's almost the end of the school year. I'm sure you can do more interesting and challenging mathematics than adding numbers, right?" The additive structure of the brick pyramid drew the students in. The fact that a question had not yet been posed seemed to add to the intrigue.

Before long, many of the students had completed the second pyramid and seemed to have noticed, with some surprise, that the top brick number (50) was larger than the one in the first pyramid (48). Without erasing the first two diagrams, I filled in a third pyramid, this time arranging the bottom numbers in this order: 1, 3, 5, 4, 2. This new order resulted in an even larger top number, namely, 61.

I challenged the class by asking, "Can you get a number higher than 61 on the top brick?" and instructed the students to work in pairs or small groups to explore, aided by templates of eight blank 5-level pyramids. After about ten minutes, during which students tried different arrangements and noticed patterns, I called on Nyree to share her group's findings. I chose Nyree because, when walking past her table, I had noticed that she had computed 61 for the top brick, yet the bottom numbers were in a different order from mine. Nyree showed her group's result: The numbers 2, 4, 5, 3, and 1 were in the bottom row, resulting in 61 on the top brick.

Displaying these two solutions next to each other on the board (1, 3, 5, 4, 2 and 2, 4, 5, 3, 1) created an opportunity for students to compare and contrast and to gain further insight into the mathematical structure of the pyramid. I suggested that students move back to their groups and explore the following questions:

- How are these two brick pyramids similar?
- How are they different?
- Why do they both yield 61 on the top?
- Can you get a number greater than 61?

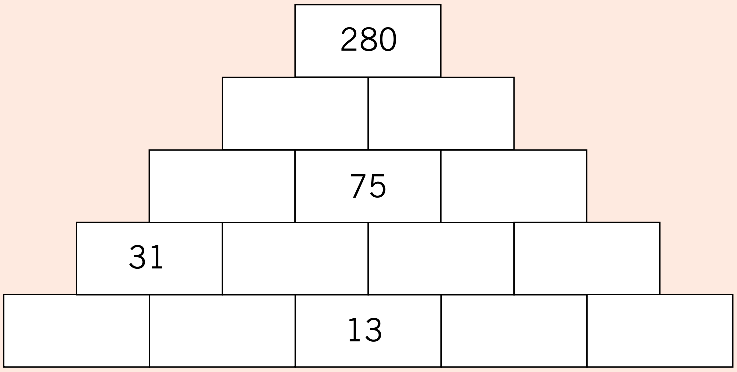
Fig. 4 Students generated five diagrams with differing degrees of abstraction and notation.

A. The necessity to shorten the expression for the utmost brick led from using addition to using multiplication as a way to express how the numbers got combined from the bottom all the way to the top of the pyramid.	1 + 4 × 3 + 6 × 5 + 4 × 4 + 2 1 + 3 + 3 + 3 + 5 + 5 + 5 + 4 3 + 5 + 5 + 5 + 4 + 4 + 4 + 2 1 + 3 + 3 + 5 3 + 5 + 5 + 4 5 + 4 + 4 + 2 1 + 3 3 + 5 5 + 4 4 + 2 1 3 5 4 2
B. This synoptic notation, which differs from the one above in that the numbers are attached to each other without operation signs, shows that the central bricks in the bottom row contribute more to the top brick than the outer ones.	1333355555544442 13335554 35554442 1335 3554 5442 13 35 54 42 1 3 5 4 2
C. This diagram, even more abbreviated than the one above, tracks all the appearances of the number 5 from bottom to top, thus emphasizing that maximizing the top number requires placing the highest number in the center bottom brick.	555555 555 555 5 55 5 5
D. This diagram, which involves the use of icons (*) instead of numbers, is a step forward in the direction of algebraic symbolizing and generalizing.	***** *** *** * ** * * * *
E. The numbers on the second to fifth rows are expressed by means of standard algebraic notation, in terms of the bottom numbers, *a, b, c, d,* and *e*. Considering the possible values of $a + 4b + 6c + 4d + e$, depending on the assignment of numbers to *a, b, c, d,* and *e*, leads toward an algebraic proof of the optimal arrangement for maximizing the top brick.	$a + 4b + 6c + 4d + e$ $a + 3b + 3c + d$ $b + 3c + 3d + e$ $a + 2b + c$ $b + 2c + d$ $c + 2d + e$ $a + b$ $b + c$ $c + d$ $d + e$ a b c d e

Students quickly noticed that (a) the largest number, 5, was in the middle of the bottom row on both pyramids; and (b) in both pyramids, the smaller numbers, 1 and 2, were in the "outside" or in the "first and last" bricks. Visibly excited, Lorena shared an explanation, "I know what's happening! You need to put the biggest numbers near the center on the bottom row . . . like . . . the 5 has to go in the middle. . . . And, also, I tried with 5 in the middle and 3 and 4 in the outside, and I got less than when I put 1 and 2 in the outside." I followed up by asking, "And why is that?" Lorena quickly responded, "It's sort of like a balance. . . . Mmm. . . . I don't know!"

Uninvited, Charles came immediately to the rescue, "I think it's because the numbers in the outside bricks don't get added as much as those in the middle." Gillian assertively added, "Yes. So we need to put 5 in the middle . . . because that's the number that gets added the most times." Students had begun to build and justify a mathematical argument and were ready to return to their groups to

Fig. 5 A brick pyramid containing fixed bricks adds an extension.



formalize those ideas in writing.

While distributing chart paper to the various groups, I asked students if they could figure out a way, using numbers, words, or symbols, to show how the numbers in the bottom bricks contribute to the sum for the top brick? While the groups worked, I circulated around the room, taking note of the various ways that students were diagramming the situation. After about ten minutes, the groups had generated the five diagrams shown in column 2 of **figure 4**. I posted all the pieces of work side by side on the blackboard. In **figure 4**, the diagrams are in ascending order, with regard to abstraction and generalization, and column 1 contains brief descriptions of the essential features of each diagram.

To frame a whole-class conversation around comparing and contrasting these five diagrams, I asked the students, "What do you notice in these pieces of work? How are these approaches similar and how are they different? What 'story' does each of these diagrams tell about the solution to this problem?" During the next five minutes, students worked individually in their notebooks, writing down observations about the five diagrams. Here are some of their comments:

> Difference between A and E: A shows what happens if the numbers in the bottom are 1, 2, 3, 4 and 5; E shows you what to do if you have any numbers: always put the biggest number in the middle and the smallest numbers in the first and the last bricks.
>
> A and E are similar because they used plus and times; but in A there are numbers and in E letters.
>
> You can tell that the group that did C paid attention mostly to the 5 to see how many times it got added up all the way to the top.
>
> C and D are nice and short but it is not easy to understand them if you don't know what the problem is about.

The last diagram, which offers a general solution using five variables (*a, b, c, d,* and *e*), represents the values for the bottom row instead of incorporating the fact that the numbers for the bottom row are consecutive. I seized this opportunity to ask the class to consider what would happen if we were to represent the numbers in the bottom row using only a single variable, such as a, $a + 1$, $a + 2$, $a + 3$, $a + 4$. That is, how would this change the expression for the top brick?

As the bell was about to ring, Cristina pointed to the coefficients in the expression $a + 4b + 6c + 4d + e$ and volunteered an observation, "I noticed something. . . . Those numbers 1, 4, 6, 4, 1 remind me of something . . . but maybe it's nothing." I asked the class (and now invite readers) to consider this question: "What do these numbers (1, 4, 6, 4, 1) remind you of and why do they show up in these brick pyramids?"

As described in the vignette above, this eighth-grade class was not directly asked how to maximize the number on the top brick. Instead, the activity was introduced by showing students one of the four arrangements of the bottom brick numbers that maximizes the top number (i.e., 1, 4, 5, 3, 2; 2, 4, 5, 3, 1; 1, 3, 5, 4, 2; and 2, 3, 5, 4, 1). While working in small groups, students produced five different diagrams for the filled-in pyramid, with differing degrees of abstraction (all the numbers, just the 5s), abbreviation (starting with sums, shifting to products), and notation (numbers, asterisks, standard symbolic-algebraic). The end of the vignette suggests a possible way in which the Brick Pyramid problem could be connected, in future lessons, to Pascal's triangle and combinations. By sharing what they noticed about similarities and differences among diagrams, students had the opportunity to "reinvent" algebra, from the bottom up, as an organizing or structuring human activity as opposed to acquiring it, top down, as inert, ready-made subject matter (Freudenthal 1991).

MORE BRICK PYRAMIDS

Brick Pyramid problems can be adapted or extended to meet the needs of and challenge all students in a given class and to create opportunities for exploring connections between mathematical ideas. For instance, how does the ordering of any five numbers (consecutive or nonconsecutive) on the bottom row determine the number on the top brick? If there are

Students had the opportunity to "reinvent" algebra, from the bottom up, as an organizing or structuring human activity.

more than five bricks in the bottom row, would that change the strategy for maximizing the number on the top brick? As an alternative, instead of trying to generate the maximum number for the top brick, we could focus on predicting its parity. In other words, given an arrangement of any five numbers for the bottom row, can you predict whether the top brick will be even or odd?

The task could also be adapted in terms of the operation that is used to build the pyramid. Addition could be changed to multiplication, so that every new brick is defined by the product of the numbers in the two bricks directly below it. If we assign a, b, c, d, and e, respectively, to the five bricks in the bottom row, the resulting expression for the product in the top brick becomes

$$a \times b^4 \times c^6 \times d^4 \times e,$$

which is similar in key ways to the $a + 4b + 6c + 4d + e$ expression from the additive pyramid. The question of how to arrange any five numbers in the bottom row so as to maximize the number on the top brick becomes focused on maximizing that expression.

Another adaptation to the Brick Pyramid problem (see Zolkower and Abrahamson 2009) is shown in **figure 5**, in which four fixed values are located on specific bricks in different rows. This time, the task is to fill in the pyramid using nonnegative integers. How many solutions can you find, and what do the solutions have in common? Why does the number 24 keep recurring? More ideas about ways to adapt or extend the brick pyramids can be found in Müller (2003).

NONROUTINE PROBLEMS

The family of Brick Pyramid problems presented here are nonroutine in the sense that their solutions are not obvious and the ways to go about describing one's thinking are not prescribed. Brick Pyramid problems are vehicles for eliciting the reinvention of algebra as a human activity that involves schematizing, generalizing, and symbolizing. This occurs in great part because the algebraic meaning is hidden, while what is visible or explicit is a spatial-numerical structure and a simple operating (addition) rule. Brick pyramids, like other such "low threshold, high ceiling" tasks, make it possible to engage all students in framing, solving, and discussing the same problem. In the hands of a teacher who can manage the expected diversity of approaches and support the sharing and exchange of ideas, such an arrangement can maximize opportunities for students to learn, not only from their teacher, but also from each other.

ACKNOWLEDGMENT

The authors wish to acknowledge Yeuk-Sze Leong for ideas on how to plan lessons around the 1, 2, 3, 4, 5 Pyramid problem and the Pyramid with Fixed Bricks problem, as well as for the opportunity of witnessing first-hand her masterful mediation of productive encounters between middle school students and Brick Pyramid problems.

REFERENCES

Freudenthal, Hans. 1991. *Revisiting Mathematics Education: China Lectures.* Dordrecht: Kluwer.

Müller, Jan Hendrick. 2003. "Exploring Number-Pyramids in the Secondary School." *The Teaching of Mathematics* 6 (1): 37–48.

McClure, Lynne. 2011. "Using Low Threshold High Ceiling Tasks in Ordinary Classrooms." http://nrich.maths.org/7701

Selter, C. 1997. "Instructional Design for Teacher Education." In *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures*, edited by M. Beshuizen, K. P. E. Gravemeijer, and E. C. D. M. van Lieshour, pp. 55–77. Utrecht: Freudenthal Institute.

Wittmann, Erich. 1995. "Mathematics Education as a 'Design Science.'" *Educational Studies in Mathematics* 29 (December): 355–74.

Zolkower, Betina, and Dor Abrahamson. "Reinventing Algebra Brick by Brick: Paradigmatic-Problematic Situations as Vehicles for Mathematizing and Didactizing." Paper presented at the Annual Meeting of the American Educational Research Association, San Diego, April 13–17, 2009.

Any thoughts on this article? Send an email to **mtms@nctm.org.**—*Ed.*

Betina A. Zolkower, betinaz@brooklyn.cuny.edu, is an associate professor at Brooklyn College, Brooklyn, New York. Her work focuses on how mathematics teachers who work in heterogeneous classrooms conduct whole-group conversations around framing and solving nonroutine problems. **Laurie H. Rubel**, lrubel@brooklyn.cuny.edu, is also an associate professor at Brooklyn College. Her research centers on diversity and equity in mathematics education.